ASYMPTOTIC DYNAMICS FOR GENERALIZED NON-LINEAR KINETIC MAXWELL-TYPE MODELS

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Maxwell-type models for non-linear kinetic equations have many applications in physics, dynamics of granular gases, economics, and other areas where statistical modeling is relevant. They model the evolution of probability measures (distribution functions) and relate to a class of non-linear space homogeneous Boltzmann equations for binary interactions where scattering probability rates of the two particles at the time of the interaction are independent of their relative velocity.

In this lecture, I have considered a large class of generalized multi-linear interacting models of Maxwell type from a rather general viewpoint, including those with arbitrary polynomial non-linearities and in any dimension space. By working in Fourier space (i.e., the space of its associated characteristic functions), it is possible to show that this class of generalized Maxwell models satisfies three fundamental properties that allow us to describe in detail the behavior of solutions depending on their initial states. In particular, it is possible to prove in the most general case an existence of self-similar solutions and study the convergence, in the sense of probability measures, of dynamically scaled solutions to the Cauchy problem to those self-similar solutions, as time goes to infinity. The properties of these self-similar solutions, which lead to Non (classical) Equilibrium Stable States (NESS), are studied in detail in [8].

More specifically, the classical elastic Boltzmann equation with Maxwell-type interactions, a mathematical model of a rarefied gas with binary collisions such that the collision frequency is independent of the velocities of colliding particles, has been well-studied in the literature (see [4], [13] and references therein). It is also well established that, due to micro reversibility (elastic, energy conservative) interactions, the model satisfies the Boltzmann $H$-theorem which yields the long time convergence of any solution of the space homogeneous initial value problem with finite energy to the Maxwell-Boltzmann probability distribution.

Maxwell-type models for granular gases were introduced relatively recently in [5] in the mathematical physics framework (see also [2] for the

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one-dimensional case and [22] and references therein for a broader discussion), and it was easy to observe that no classical $H$-theorem was available due to the micro irreversibility nature of inelastic (energy dissipative) interactions.

The motivation to study this class of models in elastic or inelastic theory was that the Maxwell-Boltzmann equation can be formulated easily in Fourier space similar to the elastic one, as done in [3] and [5], where one can study the analytical properties of the solution in the space of characteristic functions associated with probability solving the kinetic equation, that is, the Banach space of continuous, bounded functions with $L^\infty$ norms, and solutions to the spatially homogeneous inelastic Maxwell-Boltzmann equation have non-trivial self-similar asymptotics, and in addition, the corresponding self-similar solution has a power-like tail for large velocities. The latter property was conjectured in [15] and later proved in [7] and [9].

It is remarkable that such an asymptotics is absent in the elastic case with initial finite energy, due to the fact that the elastic Boltzmann equation has too many conservation laws. On the other hand, if the initial is infinite, then one can obtain self-similar asymptotics, as proven in [6] using tools from [7]. More recently, we have found exact self-similar solutions for an asymptotic model of an elastic gas of binary mixtures of Maxwell type [10]. This model, which corresponds to an energy dissipative problem of elastic colliding binary gases in the presence of a background thermostat, for a particular choice of coupling parameters, is shown to have exact solutions in Fourier space with power-like high energy tails and singular behavior at the origin, but bounded energy for all time. In particular, this result definitely suggests that self-similar asymptotics are possible for such elastic energy dissipative systems. Most recently, novel applications to diverse areas of social and economic sciences, [1], [14], [21], have introduced one-dimensional Maxwell-type models where again the self-similar asymptotics and power-like tail (Pareto) were found.

Even though all of the above discussed models describe qualitatively different processes in physics or social dynamics, their solutions have a lot in common from the mathematical point of view. It became clear as well that some further generalizations are possible, such as the inclusion in the model of multiple (not just binary) interactions, still assuming the constant (Maxwell-type) rate of interactions.

We have shown in [8] that these types of models have similar properties in the sense that there are general mathematical fixtures of Maxwell-type models, which, in turn, can explain properties of any particular model. Essentially, there is just one main result, from where one can deduce all the above discussed facts and their possible generalizations in order to establish their key properties that lead to dynamically scaled (self-similar) asymptotics.
Indeed, the Cauchy problem for generalized multi-linear Maxwell-type models, written in the Fourier space, was introduced and studied in detail [8]. For
\[ u = u(t, k) = \mathcal{F}_{v \to k}(f(t, v)) \]
the models we study can be written as integro-differential equations of the form
\[ u_t - u = \Gamma(u) \]
with \( u_0 = u(0, k) \), \( \forall t > 0 \), \( k \in \mathbb{R}^n \),
where
\[ \Gamma(u) = \sum_{l=1}^{L} \alpha_l \Gamma^{(l)}(u) \]
and
\[ \Gamma^{(l)}(u) = \int_0^\infty \cdots \int_0^\infty da_1 \cdots da_l \ A_l(a_1, \ldots, a_l) \prod_{j=1}^l u(t, a_j k), \]
under the assumption
\[ A_l(a) = A_l(a_1, \ldots, a_l) \geq 0, \quad \int_0^\infty \cdots \int_0^\infty da_1 \cdots da_l \ A(a_1, \ldots, a_l) = 1 \]
where \( A_l(a) = A_l(a_1, \ldots, a_l) \) is a generalized density of a probability measure in \( \mathbb{R}^n_+ \) for any \( l = 1, \ldots, L \). We also assume that all \( A_n(a) \) have a compact support, i.e.,
\[ A_l(a_1, \ldots, a_l) \equiv 0 \quad \text{if} \quad \sum_{j=1}^l a_j^2 > R^2, \quad l = 1, \ldots, L, \]
for sufficiently large \( 0 < R < \infty \).

When the model corresponds to the classical Boltzmann equation for binary interactions of Maxwell type, the integral operator \( \Gamma(u) \) is the Fourier transform of the \textit{Gain} part (i.e., positive contributions in probability rates) of the interaction integral, that is, \( \Gamma(u) = \mathcal{F}_{v \to k}(Q^+(f, f)(\cdot, v)) \).

Problem (0.1) is solved for \( u(t, \cdot) \) in the Banach space \( B = C_B(\mathbb{R}^n) \) with the \( L^\infty \)-norm \( \|u\| = \sup_{k \in \mathbb{R}^n} |u(k)| \), for any operator \( \Gamma \) satisfying three conditions as follows.

The most crucial concept in our approach is the notion of a class of admissible \( \Gamma \) operators, defined to be an \( \mathcal{L} \)-Lipschitz nonlinear operator, was introduced in ([8, Definition 4.1]), namely
\[ |\Gamma(u_1) - \Gamma(u_2)|(t, k) \leq \mathcal{L}((u_1) - (u_2))(t, k), \]
for some linear bounded operator \( \mathcal{L} \) defined over the Banach space \( B \).

The other two conditions are conservation of the unitary ball in the Banach space \( B \) and invariance under dilations.

It was proven ([8, Theorem 4.2]) that all generalized multi-linear Maxwell-type models under consideration, where \( \Gamma(u) \) is the Fourier transform of the positive part of the collision (i.e., \textit{Gain}) operator of the corresponding Boltzmann equation, can be written in Fourier space satisfying the evolution problem (0.1), where the three conditions on \( \Gamma(u) \) are satisfied. In this
particular case, the linear operator $\mathcal{L}$ is the linearization of $\Gamma$ about the stationary state $u = 1$.

However, these three properties of $\Gamma$ constitute the basis for the general theory.

Using just these three fundamental properties on the operator $\Gamma$, we have shown the existence and uniqueness of solutions to the initial value problem ([8, Theorem 5.2]) for $u_0 = 1 + |k|^p, p > 0$. Then, we studied the large time asymptotics under very general conditions that are fulfilled, in particular, for all of our model examples. It was shown that the $\mathcal{L}$-Lipschitz condition leads to self-similar asymptotics, provided the corresponding self-similar solution does exist, and such existence and uniqueness of self-similar solutions were proved as well ([8, Theorem 7.1]). This result can be considered, to some extent, as the main theorem for general Maxwell-type models. We established and clarified the connection to the metrics for elastic Maxwell-type problems as introduced in [17].

Further, we showed that, going back back to the original multi-linear models in probability space, one can study more specific properties of their self-similar solutions as well. We further explained how to use our theory for applications to any specific model: it is shown that the results can be expressed in terms of just one function $\mu(p) = (\lambda(p) - 1)p^{-1}, p > 0$, that depends on the spectral properties of the specific model in Fourier space related to the linear operator $\mathcal{L}$, that is, $\mathcal{L}|k|^p = \lambda(p)|k|^p, p > 0$. General properties for the solution $u$ to the initial value problem (0.1), and its anti-Fourier transformed probability measure, such as positivity, power-like tails, weak convergence of probability measures, etc., of the self-similar solutions were carefully studied and characterized following the fundamental relations between the theory probability measures and their associated characteristic functions [16], [18]. These studies also include the case of one dimensional models, where the Laplace (instead of Fourier) transform is used. Some of the necessary inequalities satisfied by the operator $\Gamma$ are typical for Fourier (Laplace) transformed Smoluchowski-type equations where the total number of particle size distribution is decreasing in time (see [19], [20] for related work).

Finally, we established in a unified statement ([8, Theorem 11.1]) the main properties of Maxwell-type models of the Boltzmann equation, corresponding to either elastic, or inelastic, or mixture gases models with or without finite initial energy, as well as applications to one-dimensional models. In addition, this result is, in particular, an essential improvement of earlier results of [6] for elastic collisions with infinity energy and quite new for the mixture model.

We remark that one may expect a probabilistic connection between Theorem 7.1 and Theorem 7.2 from [8], on the relation of rates of convergence between rates of relaxation to these self-similar states and the convergence of the corresponding Wild convolution sums formulation for a non-conservative
Maxwell model type. Such consideration may give place to results of a Central Limit Theorem type for non classical (i.e. non-Gaussian) statistical equilibrium stationary states. In the classical case of Maxwell molecules model for energy conservation with bounded initial energy, these connections has been fully established recently by [11] and [12], where, in this case, the asymptotic states are Maxwellian equilibrium distributions.

References


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