1. Introduction

These notes were written as a guideline for a short talk; hence, the references and the statements of the theorems are often not given in full details. The point of these notes is to summarize the different directions that the study of dispersive equations has taken in the last ten years. I would like to mention here a few recent textbooks that treat different parts of this subject. They could be used as supporting material and as a source of new and interesting open problems: The Nonlinear Schrödinger Equation by C. Sulem and P.-L. Sulem [28], Semilinear Schrödinger Equations by T. Cazenave [8], Global Solutions of Nonlinear Schrödinger Equations by J. Bourgain [3], and finally, the recent book of T. Tao Nonlinear Dispersive Equations: Local and Global Analysis [29].

We start with two important examples of dispersive equations. These equations were introduced as models of certain physical phenomena; for example, see [28]. Here, we are not interested in understanding their derivation, but rather in studying the quantitative and qualitative properties of their wave solutions.

Examples of Dispersive Equations

- **Nonlinear Schrödinger equation:**
  1. \( i\partial_t u + \Delta u + N(u, Du) = 0 \),
  where \( u : M \times \mathbb{R} \to \mathbb{C} \), \( M = \mathbb{R}^n \), \( \mathbb{T}^n \) or other manifolds.

- **The KdV equation:**
  2. \( \partial_t u + \partial_{xxx} u + \gamma u \partial_x u = 0 \),
  where \( u : M \times \mathbb{R} \to \mathbb{R} \), \( M = \mathbb{R}, \mathbb{T} \), and \( \gamma \in \mathbb{R} \).

  Why are these equations called dispersive?

- **Dispersion:** “Dispersive” means that the solutions of these equations are waves that spread out spatially as long as no boundary conditions are imposed. A more mathematically precise characterization is the following: Consider the general linear evolution equation

  3. \( \partial_t u + P(D)u = 0 \quad x \in \mathbb{R}^n \),
  where \( P(D) \) is a linear differential operator of symbol \( \varphi(\xi) \). Then, if one takes the space-time Fourier transform, it follows that
\[ \hat{u} (\tau - \varphi (\xi)) \mathcal{H} (\tau, \xi) = 0. \]

Hence, the Fourier transform of the solution is supported on the surface

\[ \Sigma = \{ (\tau, \xi) / \tau = \varphi (\xi) \}, \]

that lives in the phase space \( \mathbb{R} \times \mathbb{R}^n \). We give the following informal, but more mathematical definition of “dispersion”:

**Definition:** Equation 3 is “dispersive” if the surface \( \Sigma \) is “curved.”

**Examples of Non Dispersive Equations:**

- **The transport equation**

  \[ \partial_t - c \partial_x u = 0. \]

  One can easily see by taking the Fourier transform that

  \[ \hat{u} (\tau - c \xi) = 0, \]

  and that \( \Sigma = \{ (\tau, \xi) / \tau = c \xi \} \) is a line. In fact, if \( u(0) = f \), then

  \[ u(x, t) = f(x + ct); \]

  hence, if we start with a nice bump function at the origin, the bump gets translated with uniform speed \( c \) and no dispersion kicks in.

We now pass to the concept of the initial value problem (IVP):

\[
\text{IVP} = \begin{cases} 
\partial_t u + P(D) u + N(u, D^\beta u) = 0 \\
|_{t=0} = u_0 
\end{cases} 
\quad x \in M,
\]

where \( P(D) \) is a “dispersive” linear operator of degree \( \alpha \) and \( \beta < \alpha \). The function \( u_0 \) is the initial data. We usually assume that \( u_0 \in H^s(M) \), the Sobolev space of order \( s \). The following are some of the questions one would like to answer to understand the solution of the IVP and its property as a physical object:

**Question 1.** (Local and global well-posedness)

Given the initial profile \( u_0 \), does the IVP admit a unique solution \( u \in C([0, T], H^s) \)?

Is this solution stable in the \( H^s \) topology?

If yes and \( T < \infty \Rightarrow \text{local well-posedness} \)

If \( T = \infty \Rightarrow \text{global well-posedness} \).

**Question 2.** (Blow up)

If the solution is not global, which norm blows up? How does the solution look near the blow up time? What are the conditions for blow up?

**Question 3.** (Weak turbulence)

Suppose that the solution is smooth and global. How does its “energy” (\( H^1 \) norm) evolve in time? Is the “energy” moving to higher frequencies (forward cascade) or lower ones (backward cascade)?

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1 Notice that in the case of the Schrödinger equation, the surface \( \Sigma \) is a paraboloid and in the case of the KdV, \( \Sigma \) is a cubic curve!
There is one more question that we will consider in the last section of these notes. It is more abstract, but in recent years, it has received quite a lot of attention:

**Question 4.**
Do dispersive equations appear in geometry?

2. **Local well posedness: the evolution of methods**

- **The energy method:** First, one uses the equation in IVP and integration by parts to obtain a priori bounds for some norms of the solution $u$. For example, a typical bound is of the type

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \leq C(T, u_0).$$

Then, one uses, for example, the method of “finite differences” to obtain a sequence for which (*) holds and takes a weak limit to find a solution.

**Good news:** Since the method is based on integration by parts, it can be used on $\mathbb{R}^n$, $\mathbb{T}^n$, and other manifolds.

**Bad news:** In general, too many derivatives are needed; the method cannot be used in a low regularity regime.

- **The method of oscillatory integrals:** Consider the linear IVP (LIVP) and its Fourier transform

\[
\begin{aligned}
\text{LIVP} & \quad \left\{ \begin{array}{ll}
\partial_t u + P(D)u = 0 & \\
u|_{t=0} = u_0
\end{array} \right. \\
\implies & \quad \left\{ \begin{array}{ll}
\frac{d}{d\xi} \hat{u}(\xi) + i\varphi(\xi) \hat{u}(\xi) = 0 \\
\hat{u}(0, \xi) = \hat{u}_0(\xi)
\end{array} \right.
\end{aligned}
\]

Then for fixed $\xi$, it is easy to solve the above ODE and to obtain

$$\hat{u}(t, \xi) = e^{-it\varphi(\xi)}\hat{u}_0(\xi)$$

$$S(t)u_0(x) := u(t, x) = \int e^{i(x\cdot\xi - \varphi(\xi)t)}\hat{u}_0(\xi) \, d\xi.$$

If we then introduce the operator $R$, the restriction of the Fourier transform on $\Sigma = \{(\tau, \xi, \tau) = \varphi(\xi)\}$, it follows that its adjoint $R^*$ satisfies

$$S(t)u_0 = R^*u_0.$$

Then harmonic analysis (more precisely the method of stationary phases) gives estimates of the type

$$\|S(t)u_0\|_{L^q_t L^r_x} \leq \|u_0\|_{L^2_x}$$

for certain $(q, r)$. These are called **Strichartz Estimates**. These estimates are very important in order to set up a fixed point argument. In fact, by the **Duhamel Principle**, the IVP above can be proved to be equivalent to the integral equation

$$u = S(t)u_0 + i \int_0^t s(t - t')N(u)(t') \, dt'.$$
We then fix an interval $[0, T]$ and define the space

$$X^s_{[0,T]} = \left\{ \text{functions and their derivatives of order } s \text{ bounded in the appropriate Strichartz norms} \right\}.$$ 

In general, $X^s_{[0,T]} \subset C([0, T], H^s)$ is the set of continuous functions into the space $H^s$. We define

$$L(v) = S(t)u_0 + i \int_0^t S(t - t')N(v(t')) dt'$$

and prove, when possible, that

$$\|L(v)\|_{X^s_{[0,T]}} \leq C\|u_0\|_{H^s} + CT^\alpha \|v\|_{X^0_{[0,T]}}$$

$$\|L(v) - L(w)\|_{X^s_{[0,T]}} \leq cT^\alpha (\|v\|_{H^s} + \|w\|_{H^s})^{m-1} \|v - w\|_{X^s_{[0,T]}},$$

where $m$ is the order of the nonlinearity and $\alpha$ is a small positive number. If then we pick

$$\rho = 2C\|u_0\|_{H^s}, \quad T = T(\rho^{-1}) \text{ small},$$

the map $L : B(0, \rho) \to B(0, \rho)$ becomes a sharp contraction and a fixed point theorem can be used.

**Good news:** One can treat problems with much less regularity [19].

**Bad news:**

1) Since the method uses oscillatory integrals, it cannot be used for periodic problems ("oscillatory series"!)

2) The fixed point method only works when the IVP is a "small perturbation" of the corresponding LIVP. In general, this situation is guaranteed by taking either small data or short intervals of time.

**Fix to bad news 1):** Bourgain [4], [5] was able to use some basic analytic number theory to treat the periodic case!

**Fix to bad news 2):** This will be the content of the next section.

3. From local to global well-posedness

There is no doubt that proving long time existence for arbitrary large data is one of the most challenging and interesting problems in the study of dispersive equations. To start one needs to check two features associated with the IVP at hand: scaling and conservation laws.

- **Scaling:**

  To illustrate this concept, we consider a special example, namely the nonlinear Schrödinger IVP:

  \[
  IVP \begin{cases}
  i\partial_t u + \Delta u = \sigma|u|^{2p}u & x \in \mathbb{R}^n \\
  u|_{t=0} = u_0 \\
  \sigma = \pm 1.
  \end{cases}
  \]
One can easily check that
\[ u \text{ solves IVP} \iff u_\lambda(x, t) = \frac{1}{\lambda^{\frac{1}{p}}} u \left( \frac{x}{\lambda}, \frac{t}{\lambda^2} \right) \text{ solves IVP with } u_{0,\lambda} = \frac{1}{\lambda^{\frac{1}{p}}} u_0 \left( \frac{x}{\lambda} \right). \]

One has \( \|u_{0,\lambda}\|_{H^s} \simeq \lambda^{\frac{n}{2} - s - \frac{1}{p}} \|u_0\|_{H^s}. \) Based upon the sign of the exponent of \( \lambda, \) we can classify the exponent \( s \) as

**Subcritical for the IVP** if \( \frac{n}{2} - s - \frac{1}{p} < 0, \) so as \( \lambda \to +\infty, \|u_{0,\lambda}\|_{H^s} \to 0, \) (the problem is a “small perturbation of the linear one”),

**Critical for the IVP** if \( \frac{n}{2} - s - \frac{1}{p} = 0 \) (large data problem),

**Supercritical for the IVP** \( \frac{n}{2} - s - \frac{1}{p} > 0 \) (completely open!).

- **Conservation laws:**
  For the Schrödinger IVP we have

\[
\begin{align*}
1) \quad & \|u(t)\|_{L^2} = \|u_0\|_{L^2} \\
2) \quad & E(u(t)) = \frac{1}{2} \int |Du|^2 \, dx + \frac{3}{2p + 2} \int |u|^{2p+2} \, dx = E(u_0),
\end{align*}
\]

where \( E \) is the Hamiltonian.

**Remark 1:**

i) If \( \sigma = 1 \) (defocusing), then 1) + 2) \( \Rightarrow \|u(t)\|_{H^1} \leq C^*. \)

ii) If \( \sigma = -1 \) blow up. (See Weinstein [32] and Merle-Raphael [21], [22].)

Let us now consider the defocusing case \( \sigma = 1. \)

**Remark 2:** Suppose one can prove that the IVP is locally well-posed for \( u_0 \in H^s \) \( s_0 \leq s \) and \( s \) is subcritical. Then, in general, the local well-posedness is proved in an interval of time \([0, T]\) and \( T = T(\|u_0\|_{H^s}^{-1}). \) In particular, if \( s_0 \leq s \leq 1, \) using Remark 1, one could define \( T^* = T(C^*)^{-1}, \) iterate and obtain global well-posedness for \( s \geq 1! \)

- **The subcritical case and almost conservation laws:** What prevents the extension of the iteration argument described in Remark 2, below \( H^1? \) If \( u_0 \in H^s \) and \( s < 1, \) it could be that

\[ \|u(t)\|_{H^s} \to \infty \text{ very fast}. \]

If we want to iterate then at each step, we may need to shrink the time interval further by a significant amount. There is no assurance that by iteration we could cover the whole interval of time \( \mathbb{R}! \) If one wants to prove global well-posedness by iteration, one needs to
show that the increment of the norm \( \| u(t) \|_{H^s} \) of the solution \( u \) over time remains small. In the case of the \( L^2 \) and of the \( H^1 \) norms, the smallness of the increment is a consequence of the conservation laws listed above. Since we do not have a conservation law that controls the \( H^s \) norm of the solution for arbitrary \( s \), we introduce the \( I \)-operator such that
\[
\hat{If}(\xi) = m(\xi) \hat{f}(\xi)
\]
and we analyze the increment of the quantity \( E(Iu) \), where \( E \) is the Hamiltonian defined above. It is easy to see that
\[
\| u \|_{H^s}^2 \sim \| Iu \|_{H^1}^2 \leq E(Iu).
\]
Of course, one does not expect that \( E(Iu) \) is conserved since \( Iu \) is no solution anymore, but one hopes that some cancellation will still occur so that the increment of \( E(Iu) \) can be proved to be small. This is indeed the case for certain values of \( s < 1 \), and one obtains in this way global well-posedness below \( H^1 \), (see for example [11]).

- **The critical case:**
  The argument to pass from local to global presented above works when \( u_0 \in H^s \) and \( s \) is subcritical! In fact, if \( s \) is critical, local well-posedness is proved in general in \([0, T]\), with \( T = T(u_0) \)! The time depends on the profile of \( u_0 \) not only its \( H^s \) norm! Below are listed some recent results.

- **\( H^1 \) critical NLS: global well-posedness and scattering**
  * Bourgain [6]: In \( \mathbb{R}^n \), \( n = 3, 4 \), radial data in \( H^1 \).
  * Grillakis [15]: In \( \mathbb{R}^n \), \( n = 3 \) radial data in \( H^1 \), (no scattering).
  * Colliander-Keel-Staffilani-Takaoka-Tao [12]: In \( \mathbb{R}^n \), \( n = 3 \) any \( H^1 \) data.
  * Tao [30]: In \( \mathbb{R}^n \), \( n \geq 4 \) radial data in \( H^1 \).
  * Ryckman-Visan [26]: In \( \mathbb{R}^n \), \( n \geq 4 \) any \( H^1 \) data.
  * Visan [31]: In \( \mathbb{R}^n \), \( n > 4 \) any \( H^1 \) data.

- **Ingredients:** Induction, Strichartz estimates, interaction Morawetz inequality, almost conservation laws.

- **\( L^2 \) critical NLS: global well-posedness and scattering**
  * Tao-Visan: In \( \mathbb{R}^n \), \( n \geq 3 \) radial data in \( L^2 \).

- **Ingredients:** Similar to the above.

- **Big open problems:** \( L^2 \) critical NLS in \( \mathbb{R}^n \), \( n = 1, 2 \).

4. **Weak turbulence**

The notion of weak turbulence that we consider here is the migration with respect to time of the energy of the solution \( u \) to a dispersive IVP from lower
to higher frequencies. One way of conducting the study of weak turbulence is by analyzing the asymptotics of \( \|u(t)\|_{H^k} = h(t) \) for \( k \gg 1 \).

**Conjecture:**
- a) In \( \mathbb{T}^n \), \( \|u(t)\|_{H^k} \sim |t|^\alpha \) for some \( \alpha > 0 \).
- b) In \( \mathbb{R}^n \), \( \|u(t)\|_{H^*} \leq C \) \( \forall t \).

**Results:**
- For a), some results are available by Bourgain [3] and Bourgain-Kaloshin [7] (this last one is for a perturbed NLS).
- For b), some partial results, more satisfactory than the one listed above in the periodic case, are also available by Bourgain [3], Staffilani [27], and Colliander-Delort-Kenig-Staffilani [10].

Recently Colliander, Keel, Staffilani, and Tao gave an example of small initial data which evolution under a two dimension periodic cubic defocusing NLS grow as much as we please in later times. The precise statement is in the following theorem [13].

**Theorem.** Consider the two dimensional periodic, defocusing, cubic non-linear Schrödinger initial value problem. Then, for all \( K >> 1 >> \epsilon > 0 \) and \( s > 1 \), there exists a solution \( u(x,t) \) and a time \( T > 0 \) such that
\[
\|u(0)\|_{H^s} < \epsilon \quad \text{and} \quad \|u(T)\|_{H^s} > K.
\]

**Remark:** This does not prove polynomial growth, but it shows “forward cascade.”

The proof is based on a careful control of the dynamic of a finite dimension system which for certain precisely built initial data mimic exactly the behavior of the full IVP.

5. **Schrödinger maps**

We consider maps \( u : \mathbb{R}^n \rightarrow N \), where \( N \) is a Kähler manifold of dimension \( k \) with complex structure \( J \). If \( \tau(u) \) is the torsion of \( u \),
\[
\tau(u)^i = \Delta u^i + \sum_{\alpha=1}^{n} \Gamma^i_{jm}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^m}{\partial x^\alpha}, \quad i = 1, \ldots, k,
\]
then one is interested in studying the following geometric equations:
\[
\partial_t u = \tau(u) \quad \text{Harmonic map flow}
\]
\[
\partial_{tt} u = \tau(u) \quad \text{Wave map flow}
\]
\[
\partial_t u = J(u)\tau(u) \quad \text{Schrödinger map flow}.
\]

While the first two equations have been studied extensively (in particular the first one), few results are available for the last one. Below we give a short summary of what is known so far.
Existence of local smooth solution: There are several versions of this result. For example, see Ding-Wong [14], McGahagan [20], and Kenig-Pollack-Staffilani-Toro [18]. All these proofs are based on the energy method.

Global well-posedness:
- Chang–Shatah–Uhlenbeck [9]: the domain is either $\mathbb{R}^1$ or $\mathbb{R}^2$ with radial symmetry, and $N$ is a surface.
- Rodnianski-Rubenstein-Staffilani [25]: the domain is either $\mathbb{R}^1$ or $\mathbb{R}^2$ with radial symmetry, or $T$, and $N$ is a hypersurface.

There are recent and interesting developments in the more special case when the domain is $\mathbb{R}^n$ and $N = S^2$:
- Nahmod-Stefanov-Uhlenbeck [23], [24].
- Ionescu-Kenig [16]: local well-posedness in $H^s$, $s > \frac{n}{2} + \frac{1}{2}$.
- Bejenaru [1]: local well-posedness in $H^s$, $s > \frac{n}{2} + \epsilon$.
- Bejenaru [2], Ionescu-Kenig [17]: global well-posedness in $B^{s}_{2,1}$, $s = \frac{n}{7}$, and small data. (Here, $B^{s}_{2,1}$ is the Besov space of order $s$.)

REFERENCES


