WHEN DO GOOD PARAMETERIZATIONS EXIST?

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1. A BIASED HISTORIC OVERVIEW OF THE QUESTION

In 1960, Reifenberg undertook the study of the Plateau problem for $n$-dimensional surfaces in $\mathbb{R}^k$. The basic questions were: Is the infimum of the area of surfaces with a given boundary attained? What does surface mean? He considered the class

$$\mathcal{F} = \{ \Sigma \subset \mathbb{R}^k : \partial \Sigma = \Gamma, \Sigma \text{ is proper}, \Gamma \text{ is not a retraction of } \Sigma \},$$

where $\Gamma \subset \mathbb{R}^k$ is a given set homeomorphic to $S^{n-1}$. He proved:

**Theorem 1.1.** There exists $\Sigma_0 \in \mathcal{F}$ such that $\mathcal{H}^n(\Sigma_0) = \inf_{\Sigma \in \mathcal{F}} \mathcal{H}^n(\Sigma)$. Moreover for $\mathcal{H}^n$ a.e. $x \in \Sigma_0$, there exists a neighborhood of $x$ in $\Sigma_0$ which is a topological disc.

We briefly summarize the key ideas of the proof.

1. Define

$$\theta(x, r) = \frac{\mathcal{H}^n(\Sigma \cap B(x, r))}{\omega_n r^n},$$

$$\beta(x, r) = \frac{1}{r} \inf D[\Sigma \cap B(x, r), L \cap B(x, r)],$$

where the infimum is taken over all $n$ planes $L$ through $x$. Here, $D$ denotes the Hausdorff distance between 2 sets. Recall $A, B$ closed sets in $\mathbb{R}^k$

$$D[A, B] = \sup_{z \in B} \{ \text{dist}(z, A) \} + \sup_{z \in A} \{ \text{dist}(z, B) \}.$$ 

If $\beta(x, r) \leq \delta$, there exists an $n$-plane $L(x, r)$ containing $x \in \Sigma$ and such that

- $\Sigma \cap B(x, r) \subset (L(x, r) \cap B(x, r), \delta r)$

$\delta r$–neighborhood of $L(x, r) \cap B(x, r)$
(2) Monotonicity of the density ratio $\theta(x, r)$: If $\Sigma$ is a minimizer then for each $x$, $\theta(x, r)$ is an increasing function of $r$, and for a.e. $x$,

$$\theta(x) = \lim_{r \to 0} \theta(x, r) \geq 1.$$  

(3) Given $\delta > 0$, there exists $\epsilon > 0$ such that if \n
$$1 \leq \theta(x_0, 2r_0) < 1 + \epsilon,$$  

then $\beta(x, r) \leq \delta$ for $B(x, r) \subset B(x_0, r_0)$.

(4) Topological disc property:

**Theorem 1.2** ([18]). There exists $\delta > 0$ such that if

$$\beta(x, r) \leq \delta$$  

for $x \in \Sigma$ and $r < 1$, then locally $\Sigma$ is an $n$ dimensional topological disc. In particular, there exists $\gamma > 0$ such that $\Sigma$ is a $C^{0, \gamma}$ $n$-dimensional submanifold.

**Example 1.3.** The following sets are well approximated by planes in the sense that $\beta(x, r)$ is small for $r$ small; in fact, $\beta(x, r)$ tends to 0 as $r$ approaches 0.

(1) If $\Sigma$ is a $C^1$ $n$-submanifold, then for $x \in \Sigma$,

$$\lim_{r \to 0} \beta(x, r) = 0.$$
(2) If
\[ \Sigma = \{ x = (x_1, x_2) : x_2 = \sum_{k=1}^{\infty} \frac{\cos(2^k x_1)}{2^k \sqrt{k}} \}, \]
then for \( x \in \Sigma \),
\[ \lim_{r \to 0} \beta(x, r) = 0. \]
These examples are very different in nature. In particular, the second set, which is the graph of an \( \alpha \)-Zygmund function, provides an example of a set whose boundary has tangent lines almost nowhere. This set has Hausdorff dimension 1, but the 1-dimensional Hausdorff measure restricted to the set is locally infinite.

Reifenberg’s topological disc property result was somewhat overshadowed by his subsequent work. In 1964, he proved the epiperimetric inequality for area minimizers. This inequality yields a rate of decay for \( \theta(x, r) \) toward 1 as \( r \) tends to 0. This implies that for a.e. \( x \in \Sigma \) and for \( r > 0 \) small enough, \( \beta(x, r) \leq r^{2\alpha} \), which is enough to ensure that locally \( \Sigma \) is a \( C^{1,\alpha} \) submanifold. A standard argument shows that \( C^{1,\alpha} \) minimizers are real analytic [19], [20].

Over the last 20 years, the ideas involved in the proof of the topological disc property, as well as its content, have been used to formulate criteria that ensure the existence of good parameterizations. We include below some examples of results whose proofs require the use of Reifenberg’s ideas.

**Example 1.4.** (1) The singular set of an energy minimizing harmonic map from \( B^4 \) into \( S^2 \) is (locally) a finite set and a finite union of Hölder continuous curves [8].

(2) Criteria for existence on biLipschitz parameterizations for subsets of Euclidean space:

**Theorem 1.5** ([21]). If
\[ \int_0^1 \frac{\beta^2(r)}{r} \, dr < \infty, \]
where \( \beta(r) = \sup \{ \beta(x, r) : x \in \Sigma \} \) then locally \( \Sigma \) admits biLipschitz parameterizations.

(3) Construction of snowballs [7].

2. The density ratio as an indicator of regularity

**Question 2.1.** For \( \Sigma \subset \mathbb{R}^k \), does the density ratio
\[ \theta(x, r) = \frac{\mathcal{H}^n(\Sigma \cap B(x, r))}{\omega_n r^n} \]
provide any information about the structure and/or the regularity of \( \Sigma \)?
Besicovitch was the first one to consider this question (see [1], [2], [3], [4], [5]). Marstrand (see [10], [11], [12], [13]), Mattila (see [14], [15]), and Preiss (see [16]) have also made substantial contributions to this fundamental area of Geometric Measure Theory. These four authors have provided most of the tools used to address this type of problem.

**Theorem 2.2 ([1]).** If \( n = 1, \ k = 2 \) and for a.e. \( x \in \Sigma \),
\[
\theta(x) = \lim_{r \to 0} \theta(x, r) = 1,
\]
then \( \Sigma \) is 1-rectifiable.

**Theorem 2.3 ([16]).** If for a.e. \( x \in \Sigma \)
\[
\theta(x) = \lim_{r \to 0} \theta(x, r) = 1,
\]
then \( \Sigma \) is n-rectifiable.

Recall that a set is n-rectifiable if it can be written as the countable union of Lipschitz images of \( \mathbb{R}^n \) union a set of n-Hausdorff measure 0. A key step in the proof of the two theorems above is the study of the tangent sets and tangent measures to \( \Sigma \) and \( \mathcal{H}^n \mathcal{L} \Sigma \) (the n-dimensional measure restricted to \( \Sigma \)), respectively. These tangent objects are obtained as limits of blow-up sequences. At almost every point in \( \Sigma \) the tangent measures are n-uniform. A measure \( \mu \) is n-uniform if for every \( q \) in the support of \( \mu \) and every radius \( r > 0 \)
\[
\mu(B(q, r)) = Cr^n
\]
for a fixed constant \( C > 0 \).

**Theorem 2.4 ([9]).** Let \( \Sigma \subset \mathbb{R}^{n+1} \) be such that for \( x \in \Sigma \) and \( r > 0 \)
\[
\theta(x, r) = 1,
\]
i.e., \( \mathcal{H}^n(\Sigma \cap B(x, r)) = \omega_n r^n \).
Then modulo translation and rotation either
- \( \Sigma = \mathbb{R}^n \times \{0\} \) or
- if \( n \geq 3 \), \( \Sigma = \{ x \in \mathbb{R}^{n+1} : x_1^2 = x_2^2 + x_3^2 \} \).

**Theorem 2.5 ([16], [6]).** There exists \( \delta > 0 \) such that for \( \Sigma \subset \mathbb{R}^k \) satisfying
\[
\theta(x, r) = 1, \quad \text{for } x \in \Sigma, \quad r > 0 \quad \text{and} \quad \beta(x, r) < \delta \quad \text{for} \quad r > 1,
\]
then \( \Sigma \) is an n-plane.

**Theorem 2.6 ([6]).** There exists \( \delta > 0 \) such that if \( \Sigma \subset \mathbb{R}^{n+1} \) satisfies
\[
\theta(x, r) = 1, \quad \text{for } x \in \Sigma, \quad r > 0 \quad \text{and} \quad \beta(x, r) < \delta \quad \text{for} \quad r < 1,
\]
then \( \Sigma \) is a smooth n-dimensional submanifold of \( \mathbb{R}^{n+1} \).

**Remark.** For \( n = 2 \), Kowalski proved that if \( \Sigma \) is as above then \( \Sigma \) is a collection of disjoint spheres.

Recall that Reifenberg’s epiperimetric inequality provides a decay rate of the density ratio toward 1 as \( r \) tends to 0. For area minimizers, this is enough to ensure regularity. We now address this question for general sets.
Theorem 2.7 ([6]). There exists $\delta > 0$ such that if $\Sigma \subset \mathbb{R}^{n+1}$ satisfies
$$|\theta(x, r) - 1| \leq Cr^\alpha \quad \text{for} \quad x \in \Sigma, \ r < 1,$$
for some $\alpha \in (0, 1)$ and
$$\beta(x, r) < \delta \quad \text{for} \quad x \in \Sigma, \ r < 1,$$then $\Sigma$ is a $C^{1,\gamma}$ $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. Here $\gamma$ depends on $\alpha$ and $n$.

Theorem 2.8 ([17]). There exists $\delta > 0$ such that if $\Sigma \subset \mathbb{R}^k$ satisfies
$$|\theta(x, r) - 1| \leq Cr^\alpha \quad \text{for} \quad x \in \Sigma, \ r < 1,$$
for some $\alpha \in (0, 1)$ and
$$\beta(x, r) < \delta \quad \text{for} \quad x \in \Sigma, \ r < 1,$$then $\Sigma$ is a $C^{1,\gamma}$ $n$-dimensional submanifold of $\mathbb{R}^k$. Here $\gamma$ depends on $\alpha$, $n$, and $k$.

These two results require a detailed study of the interplay between the first and second moments as defined in [9]. In the high codimension case, a multiscale analysis is needed to produce normal spaces at each point.

3. Open questions

The following two questions partially motivated the last two results. They remain open.

(1) If $\Sigma \subset \mathbb{R}^{n+1}$ satisfies
$$|\theta(x, r) - 1| \leq Cr^\alpha \quad \text{for} \quad x \in \Sigma, \ r < 1,$$can $\Sigma$ be locally parameterized as a $C^{1,\gamma}$ image of either the unit ball in $\mathbb{R}^n$ or $\{x \in \mathbb{R}^{n+1} : |x| < 1 \text{ and } x_4^2 = x_1^2 + x_2^2 + x_3^2\}$?

(2) Let $\Sigma \subset \mathbb{R}^k$ be such that for $x \in \Sigma$ and $r > 0$
$$\theta(x, r) = 1.$$Describe $\Sigma$.

References


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