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OLGA LADYZHENSKAYA AND OLGA OLEINIK: TWO GREAT WOMEN MATHEMATICIANS OF THE 20TH CENTURY

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This short article celebrates the contributions of women to partial differential equations and their applications. Although many women have made important contributions to this field, we have seen the recent deaths of two of the brightest stars–Olga Ladyzhenskaya and Olga Oleinik–and in their memory, we focus on their work and their lives.

The two Olgas had much in common and were also very different. Both were born in the 1920s in the Soviet Union, grew up during very difficult years and survived the awful death and destruction of the second world war. Shortly after the war, they were students together at Moscow State University where they were both advised by I. G. Petrovsky, whose influence on Moscow mathematics at the time was unsurpassed. Both were much influenced by the famous seminar of I. M. Gelfand, and both young women received challenging problems in PDE from Gelfand. In 1947, both Olgas graduated from Moscow State University, and then their paths diverged. Olga Oleinik remained in Moscow and continued to be supervised by Petrovsky. Her whole career was based in Moscow; after receiving her Ph.D. in 1954, she became first a professor and ultimately the head of the Department of Differential Equations at Moscow State University. Olga Ladyzhenskaya moved in 1947 to Leningrad, and her career developed at the Steklov Institute there. Like Oleinik, her mathematical achievements were very influential; as a result of her work, Ladyzhenskaya overcame discrimination to become the uncontested leader of the Leningrad school of PDE.

It is our understanding that the personalities of the two Olgas were rather different, although they were both women of great strength and determination. Oleinik was a member of the academic establishment with all that this implied in the Soviet system, while Ladyzhenskaya, whose father was arrested and killed as a “class enemy,” was outside the establishment and at times openly critical of the system. However, in both cases, their superb mathematics well merited the ultimate seal of approval of the establishment, namely election to membership in the Russian Academy of Sciences.

FIGURE 1. Olga Ladyzhenskaya

OLGA LADYZHENSKAYA

Olga Alexandrovna Ladyzhenskaya was born on March 7, 1922, in the rural Russian town of Kologriv and died in her sleep on January 12, 2004, in St. Petersburg, Russia, at the age of 81. She left a wonderful legacy for mathematics in terms of her fundamental results connected with partial differential equations and her school of students, collaborators, and colleagues in Russia. In a life dedicated to mathematics, she overcame personal tragedy arising from the cataclysmic events of 20th century Russia to become one of that country's leading mathematicians.

In 1939, she passed the entrance exams for Leningrad University, which at the time was the best university in the Soviet Union. However, she was denied a place as an undergraduate at the university because despite being an exceptionally gifted young woman, she was one whose father disappeared in Stalin's gulag. Her father had taught mathematics at a high school, and it was her father who introduced Olga at an early age to mathematics and calculus. In 1937, her father was arrested and later killed by the NKVD, the forerunner of the KGB. Life then became extremely difficult for his family who lived in disgrace and poverty as the family of a class enemy. With help from friends, Olga finally became a student at Moscow State University in 1943, and she graduated in 1947. There I. G. Petrovsky was her advisor, and she was also strongly influenced by I. M. Gelfand.

Ladyzhenskaya married Leningrad mathematician A. A. Kiselev in 1947 and became a graduate student at Leningrad State University. Her advisors were S. L. Sobolev and V. I. Smirnov. Her Ph.D. thesis, defended in 1949, was a breakthrough in the theory of PDE, and later developments concerning weak solutions to initial boundary value problems became important concepts in mathematical physics. From 1947 on, she was very actively involved in the Leningrad seminar on mathematical physics that brought together many mathematicians working in PDE and their applications. She remained one of the leaders of the seminar until her death.

For most of her professional career, Ladyzhenskaya was a member of the Steklov Institute in Leningrad/St Petersburg (called LOMI and now called POMI). She rose to become one of the most distinguished and influential members of POMI. She was elected to the Russian National Academy of Sciences (as corresponding member in 1981 and as full member in 1990). Among her prizes was the Kovaleskaya Prize of the Russian Academy. Her mathematical achievements were honored in many countries. She was a foreign member of several academies including the Leopoldina, the oldest German academy. Among other offices, she was President of the Mathematical Society of St. Petersburg and as such, a successor of Euler. Recently, she was awarded the degree of Doctoris Honoris Causa by the University of Bonn, and an excellent description of her achievements may be found in the *laudatio* given for this occasion by M. Struwe [7].

Ladyzhenskaya made deep and important contributions to the whole spectrum of partial differential equations and worked on topics that ranged from uniqueness of solutions of PDE to convergence of Fourier series and finite difference approximation of solutions. She used functional analytic techniques to treat nonlinear problems using Leray-Schauder degree theory and pioneered the theory of attractors for dissipative equations. Developing ideas of De Giorgi and Nash, Ladyzhenskaya and her coauthors gave the complete answer to Hilbert's 19th problem concerning the dependence of the regularity of the solution on the regularity of the data for a large class of second order elliptic and parabolic PDE.

She published more than 250 articles and authored or co-authored seven monographs and textbooks. Her very influential book *The Mathematical Theory of Viscous Incompressible Flow*, published in 1961, has become a classic in the field. Her main mathematical love was the PDE of fluid dynamics, particularly the Navier-Stokes equation. This equation has a long and glorious history but remains extremely challenging: for example, the issue of existence of physically reasonable solutions to the Navier-Stokes equations in three dimensions was chosen as one of the seven "million dollar" prize problems of the new millennium by the Clay Mathematical Institute (for details, see the problem description by Fefferman [2]). The three-dimensional problem remains open to this day, although it was in the 1950s that Ladyzhenskaya obtained the key result of global unique solvability of the initial boundary problem for the two-dimensional Navier-Stokes equation. She continued to obtain influential results and raise stimulating issues in fluid dynamics, even up to the days before her death.

Ladyzhenskaya also considered fluid dynamics outside the framework of the Navier-Stokes equations. She explored alternative models for such challenging issues as turbulence, and this led her to study the notion of an attractor for infinite dimensional dynamical systems. In this connection, she opened a new direction in the theory of PDE, namely "stability in the large." Further details concerning Ladyzhenskaya's significant mathematical achievements may be found in the memorial article in the *Notices of the AMS* [3] and in the volumes published in honor of her 80th birthday [1].

Ladyzhenskaya was a woman of great charm and beauty. She was part of a circle of Russian intellectuals of world-wide fame including A. Solzhenitsyn, A. Akhmatova, and J. Brodsky. G. Seregin and N. Uraltseva, her friends, colleagues and collaborators, tell us that it was not only Olga's scientific results, though truly deep and fundamental, but also her personal integrity and energy that played a special role in her contribution to mathematics.

FIGURE 2. Olga Ladyzhenskaya (with Tamara Rozhkovskaya in mirror)

Selected honors and publications of Olga Ladyzhenskaya

1969 The State Prize of the USSR
1985 Elected a foreign member of the Deustche Akademie Leopoldina
1989 Elected a member of the Accademia Nazionale dei Lincei
1990 Elected a full member of the Russian Academy of Sciences
2002 Awarded the Great Gold Lomonosov Medal of the Russian Academy
2002 Doctoris Honoris Causa, University of Bonn

with N. Uraltseva, *Linear and Quasilinear Elliptic Equations*. Moscow: Izdat. "Nauka," 1964; Engl. trans., New York-London: Academic Press, 1968.

The Mathematical Theory of Viscous Incompressible Flow. Moscow: Gosudarstv. Izdat. Fiz.-Mat. Lit., 1961; Engl. trans., New York-London-Paris: Gordon and Breach, Science Publishers, 1969.

with A. Kiselev, "On the existence and uniqueness of the solution to the nonstationary problem for a viscous, incompressible fluid," (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 665–680.

"Solution 'in the large' of the nonstationary boundary value problem for the Navier-Stokes system with two space variables," Comm. Pure Appl. Math. **12** (1959), 427–433.

with V. A. Solonnikov and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*. Moscow: Izdat. "Nauka," 1968; Engl. trans., Translations of Mathematical Monographs, Vol. 23. Providence, R.I.: American Mathematical Society, 1967.

Attractors for Semigroups and Evolution Equations. Lezioni Lincee. Cambridge: Cambridge University Press, 1991.

Olga Oleinik

Olga Arsenievna Oleinik was born in the Ukraine on July 2, 1925, and died of cancer on October 13, 2001.

She obtained her Ph.D. from Moscow State University (where she spent her career) in 1954, a student of Ivan Petrovsky, one of the founders of the modern theory of partial differential equations (PDE). As Petrovsky's successor, she built a strong team in PDE, and from the start of her career, she also explored applications in elasticity and in several aspects of fluid flow, including compressible gas dynamics and the filtration equation of flow in porous media.

Near the beginning of her career, she contributed greatly to the theory of hyperbolic conservation laws, then in its infancy. Conservation laws are nonlinear partial differential equations of the form

$$ut + f(u)x = 0. \tag{1}$$

FIGURE 3. Olga Oleinik

Here, u is a scalar or vector quantity, and f is a corresponding flux function. Equation (1) expresses conservation of the components of u–typically mass, momentum, and energy. The system is hyperbolic when the Jacobian of the flux, df, has a full set of real eigenvalues and eigenvectors. When $f(u) = Au$ for a matrix A, the system is linear and its solutions, including weak solutions, are well understood from linear theory.

The wartime work of Courant, Friedrichs, and others had established the necessity of finding a nonlinear theory for weak solutions, as classical hyperbolic theory could not explain the spontaneous formation of shocks, the fact that nonlinear equations gave rise to discontinuities that did not propagate along characteristics, or the ensuing questions about lack of uniqueness. In addition, global existence theorems were lacking, and even the correct function spaces in which to seek solutions were unknown, despite the fact that these equations underlay the technology of explosions and the new field of supersonic flight. The work of Oleinik changed this. She proved existence of weak solutions to the scalar equation (1), for general flux functions, showing they were limits of the perturbed equation

$$ut + f(u)x = euxx, \qquad (2)$$

generalizing work of Hopf. It would be almost 50 years until this result was broadened to systems of conservation laws. In her investigation, Oleinik found the correct space–BV–for solutions. She also developed what is now called the Oleinik entropy condition for uniqueness of solutions of the scalar equation (1). Finally, she proved a uniqueness result for solutions of certain systems, modeled on gas dynamics–this at a time when no existence theorems for systems had yet been proved; the first existence theorem for systems, due to Glimm, appeared shortly after her result. Again, the uniqueness result was not improved for over 30 years.

Oleinik developed fundamental mathematical results in other areas related to classical fluid flow: boundary layer theory (the stability of boundary layers, where viscosity is important only close to the body) and degenerate elliptic equations (motivated by change of type in steady transonic flow). In this last field, termed "equations with non negative characteristic form," she completed and extended work of the Italian school, notably Fichera and Tricomi.

Later in her career, Oleinik turned her attention to diverse other areas: the Stefan problem, in which the mathematical interest is that it provides a free boundary problem for a parabolic equation and the applications interest is in phase transitions. She also provided the basic theory of weak solutions for the nonlinear degenerate parabolic equation, known as the filtration equation. In the 1990s, Oleinik, with Jikov and Kozlov, helped to develop the mathematical theory of homogenization.

In all, her list of publications indexed by Math Reviews includes over 400 items, displaying an astonishing breadth and depth. A memorial article in the *Notices* [4] remarks on her love of travel, her eagerness to make contacts between Soviet and Western mathematicians, and her loyalty to her friends.

FIGURE 4. Cora Sadoksy and Olga Oleinik after Oleinik's Noether Lecture

Selected honors and publications of Olga Oleinik

1981 Honorary Doctorate, University of Rome
1983 Elected an Honorary Member of the Royal Society of Edinburgh
1988 Elected a member of the Accademia Nazionale dei Lincei
1990 Elected a full member of the Russian Academy of Sciences
1996 Association for Women in Mathematics Noether Lecturer

She was also awarded the Petrovsky Prize and the Medal of the Collège de France.

"Discontinuous solutions of non-linear differential equations," Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 3–73.

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Figure 1: Reproduced with permission from *Nonlinear Problems in Mathematical Physics and Related Topics. In Honor of Professor O. A. Ladyzhenskaya I, II.* Ed. M. Sh. Birman, S. Hildebrandt, V. A. Solonnikov, and N. N. Uraltseva. International Mathematical Series. Kluwer/Plenum Publishers (English; now Springer) and Tamara Rozhkovskaya (publisher, Russian).

Figure 2: Tamara Rozhkovskaya

Figure 3: From the collection of E. Radkevich

Figure 4: Dawn Wheeler, AWM

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