A remarkable feature of dissipative partial differential equations (PDEs) is the existence of a global attractor to which all the solutions converge as time goes to infinity. The global attractor is the minimal closed set in the phase space $H$ (i.e., the fundamental space in which the solutions exist), which uniformly attracts the trajectories starting from any a priori given bounded set in $H$. However, the fact that a PDE is dissipative may not automatically ensure the existence of a global attractor. For instance, the 2D Navier-Stokes equations (NSE) on a bounded domain $\Omega \subset \mathbb{R}$, when supplemented with appropriate boundary conditions, possess a global attractor, but it is not yet known whether this holds for the 3D NSE.

Nevertheless, even for the 3D NSE, one can prove that there exists a minimal weakly closed set in $H$ for which the attracting property holds with respect to the weak topology in $H$ [7]. Therefore, this attractor is referred to as the weak global attractor. For convenience, the first attractor mentioned above will be called the strong global attractor. When the strong global attractor is compact in $H$ (e.g., in the 2D NSE), then it is also the weak global attractor. But, in any case, the weak global attractor is an appropriate generalization of the strong global attractor since it captures the long-time behavior of the solutions. In particular, the support of any time-average measure of the 3D NSE is included in the weak global attractor.

The first obstacle in studying the long-time behavior of the 3D NSE is the fact that the semigroup of solution operators cannot be defined due to the lack of uniqueness proof. There are several abstract frameworks for studying dynamical systems without uniqueness. In one approach, used by Babin & Vishik, and which goes all the way back to a work by Barbashin, a trajectory is a function of time with values in the set of all subsets of a phase space. In [5], we define an evolutionary system $E$, which is closer to Ball’s generalized semiflow $G$ [2], where a trajectory is a function of time with values in the phase space, and there may be more than one trajectory with given initial data. Since the definition of $E$ does not contain the hypotheses of concatenation and upper semicontinuity with respect to initial data, the Leray-Hopf weak solutions of the 3D NSE form an evolutionary system.
To define an evolutionary system, first let \((X, d_s(\cdot, \cdot))\) be a metric space endowed with a metric \(d_s\), which will be referred to as a strong metric. Let \(d_w(\cdot, \cdot)\) be another (weak) metric on \(X\) satisfying the following conditions:

1. \(X\) is \(d_w\)-compact.
2. If \(d_s(u_n, v_n) \to 0\) as \(n \to \infty\) for some \(u_n, v_n \in X\), then \(d_w(u_n, v_n) \to 0\) as \(n \to \infty\).

In the case of the 3D NSE, the space \(X\) can be defined as a closed absorbing ball in the phase space. Let

\[ T := \{ I : I = [T, \infty) \subset \mathbb{R}, \text{ or } I = (-\infty, \infty) \}, \]

and for each \(I \subset T\) let \(F(I)\) denote the set of all \(X\)-valued functions on \(I\).

**Definition 1.** A map \(E\) that associates to each \(I \in T\) a subset \(E(I) \subset F\) will be called an evolutionary system if the following conditions are satisfied:

1. \(E([0, \infty)) \neq \emptyset\).
2. \(E(I + s) = \{ u(\cdot) : u(\cdot - s) \in E(I) \}\) for all \(s \in \mathbb{R}\).
3. \(\{ u(\cdot)|_{I_2} : u(\cdot) \in E(I_1) \}\) \(\subset E(I_2)\) for all pairs of \(I_1, I_2 \in T\), such that \(I_2 \subset I_1\).
4. \(E((\infty, \infty)) = \{ u(\cdot) : u(\cdot)[T, \infty) \in E([T, \infty)) \}\) for all \(T \in \mathbb{R}\).

A set \(E(I)\) is referred to as the set of all trajectories on the time interval \(I\); trajectories in \(E((\infty, \infty))\) are called complete. Let

\[ R(t)A := \{ u(t) : u \in A, u(\cdot) \in E([0, \infty)), A \subset X \}. \]

For a set \(A \subset X\) and \(r > 0\) denote \(B_\bullet(A, r) = \{ u : d_\bullet(u, A) < r \}\), where

\[ d_\bullet(u, A) := \inf_{x \in A} d_\bullet(u, x), \quad \bullet = s, w. \]

A set \(A \subset X\) uniformly attracts a set \(B \subset X\) in \(d_\bullet\)-metric if for any \(\epsilon > 0\) there exists \(t_0\), such that

\[ R(t)B \subset B_\bullet(A, \epsilon), \quad \text{for all } t \geq t_0. \]

**Definition 2.** A set \(A \subset X\) is a \(d_\bullet\)-attracting set (\(\bullet = s, w\)) if it uniformly attracts \(X\) in \(d_\bullet\)-metric.

**Definition 3.** A set \(A_\bullet \subset X\) is a \(d_\bullet\)-global attractor (\(\bullet = s, w\)) if \(A_\bullet\) is a minimal \(d_\bullet\)-closed \(d_\bullet\)-attracting set.

Note that the definition of the evolutionary system \(E\) already exploits the effect of dissipativity, namely the existence of an absorbing ball, and a global attractor is defined as the minimal closed attracting set in the corresponding topology. We show that every evolutionary system always possesses a weak global attractor; moreover, if the strong global attractor exists and is weakly closed, then it has to coincide with the weak global attractor. Note that some classical definitions require a global attractor to be an invariant set.
We note that under a condition which is, for example, satisfied by the Leray-Hopf weak solutions of the 3D NSE, the weak global attractor is also the maximal bounded invariant set.

It is known that if a weak global attractor for the 3D NSE is bounded in $H^1$, then it is in fact strong. Moreover, Ball [2] showed that if a generalized semiflow for a dissipative evolutionary system is asymptotically compact, then a strong global attractor exists. This implies that the strong global attractor for the 3D NSE exists under the condition that all weak solutions are strongly continuous from $(0, \infty)$ to $L^2$ (see [2]). In [5], we show that even without the assumptions of concatenation and upper semicontinuity with respect to the initial data, the asymptotic compactness implies that the weak global attractor is the minimal compact attracting set in the strong metric; i.e., the weak global attractor is, in fact, the strong compact global attractor. Applied to the 3D NSE, this result implies the existence of a strong compact global attractor in the case when solutions on the weak global attractor are continuous in $L^2$. (See also [8] for a similar result.)

In [4], we further investigate properties of the evolutionary system $\mathcal{E}$, focusing on omega-limits and attracting, invariant, and quasi-invariant sets. Among other things, we derive weak and strong tracking properties and study a trajectory attractor of an evolutionary system. In addition, a rather general theorem on the existence of the strong global attractor is proved. Under some conditions that are, for example, satisfied by the Leray-Hopf weak solutions to the 3D NDE, the theorem yields that $A_s$ exists and coincides with $A_w$, provided that all the complete trajectories are strongly continuous. See [6] for one implication of this theorem, which is the existence of a strong global attractor for the inviscid dyadic model of fluid equations. This surprising fact is a result of a self-dissipation mechanism due to the loss of regularity of solutions. See also [3] for an application of this theorem to a viscous dyadic model.

References


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