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FULLY IMPLICIT DISCONTINUOUS GALERKIN SCHEME FOR TWO-PHASE FLOW

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The understanding of multiphase flow is of crucial importance to agencies concerned with energy, in particular oil production. This work deals with the modeling of two-phase flow, for example the flow of a wetting phase (such as water) and a non-wetting phase (such as dense non-aqueous phase liquids), in a porous medium with possibly heterogeneous characteristics. This type of flow is mathematically modeled by a nonlinear system of coupled partial differential equations (PDEs) that express the conservation laws of mass and momentum and that in general can only be solved by the use of numerical methods [3]. In this work we consider implicit pressure-saturation formulation for two-phase flow. The primary variables are the pressure of the wetting phase and the saturation of the non-wetting phase. They are approximated by discontinuous polynomials of different degrees. The resulting finite dimensional problem is an algebraic system of nonlinear equations to which the Newton-Rapshon iterative scheme is applied.

Let Ω be a polygonal porous medium in $\mathbb{R}^2$. The formulation of the model for the coupled pressure-saturation equations for incompressible two-phase flow with unknowns p_w and s_n is given by:

$$-\nabla \cdot (\lambda_t K \nabla p_w + \lambda_n K \nabla p_c) = q_w + q_n, \tag{1}$$

$$-\phi \frac{\partial s_n}{\partial t} - \nabla \cdot (\lambda_w K \nabla p_w) = q_w. \tag{2}$$

- Permeability $\mathbf{K}$, symmetric positive definite tensor, can be discontinuous in the space variable.
- λ_n and λ_ω are oil and water mobilities $\lambda_\alpha = \frac{\kappa_{r\alpha}}{\mu_\alpha}$.
- For equations (1) and (2), we use the Brooks-Corey model

  $$\kappa_{r\omega}(s_n) = (1 - s_n)^{\frac{2+3\theta}{\theta}}, \qquad \kappa_{rn} = s_n^2(1 - (1 - s_n)^{\frac{2+\theta}{\theta}}),$$

  where parameter $\theta \in [0.2, 3.0]$ is the characteristic of the inhomogeneity of the medium.

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- $p_c = p_n - p_w$ is the capillary pressure. For the Brooks-Corey model, we have
$$p_c = p_d(1 - s_n)^{-\frac{1}{\theta}},$$
where p_d is the entry pressure.
- μ_n, μ_w, and ϕ are the phase viscosities and the porosity.
- An additional closure relation holds: $s_\omega + s_n = 1$.

This system of partial differential equations can be classified as mixed hyperbolic-parabolic type. In this work, we introduce fully implicit, fully coupled **Nonsymmetric Interior Penalty Galerkin Method** (NIPG) in space and **Backward Euler Method** for the time discretization [1], [2], [4]. No slope limiters or upwinding is used. Numerical solution of the quarter-five spot problem with heterogeneous media is presented.

The domain is embedded in the square $(0, 100)^2$; an injection well is located at the bottom left corner of the domain with $p^-_{dir} = 3.5 \times 10^5 Pa$, and a production well is located at the top right corner of the domain with $p^+_{dir} = 1.5 \times 10^5 Pa$. No flow boundary condition is assumed on the rest of the boundary.

The unstructured triangular mesh consisting of 66 triangles is given in Figure 1. Discontinuous polynomials of degree $r_p = 5, r_s = 3$ are used to approximate pressure of the wetting phase and saturation of the non-wetting phase, respectively.

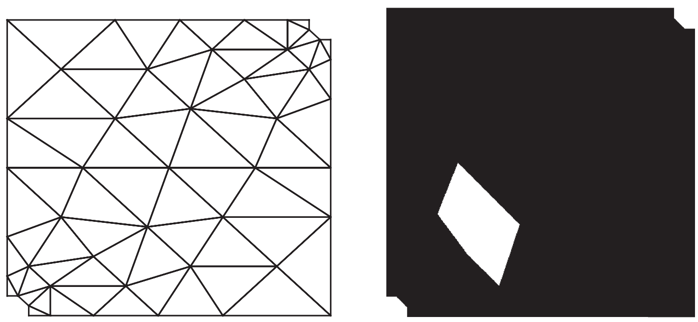
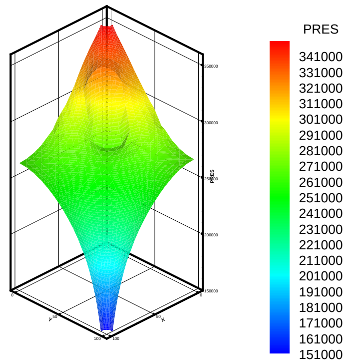


FIGURE 1. Unstructured triangular mesh (left) and permeability field (right): $k = 10^{-12}m^2$ in white regions and $k = 10^{-8}m^2$ in rest of domain.

The entry pressure for the capillary pressure is $p_d = 3 \times 10^3 Pa$ and the Brooks-Corey parameter is $\theta = 3$. The viscosities are $\mu_w = 5 \times 10^{-4}$ and $\mu_n = 2 \times 10^{-3}$. The porosity is $\phi = 0.2$. The permeability field $K = kI$ is discontinuous with $k = 1 \times 10^{-8}m^2$ everywhere except in an inclusion where $k = 1 \times 10^{-12}m^2$ (Figure 1). Numerical solutions for the wetting phase pressure and saturation are presented in Figure 2.



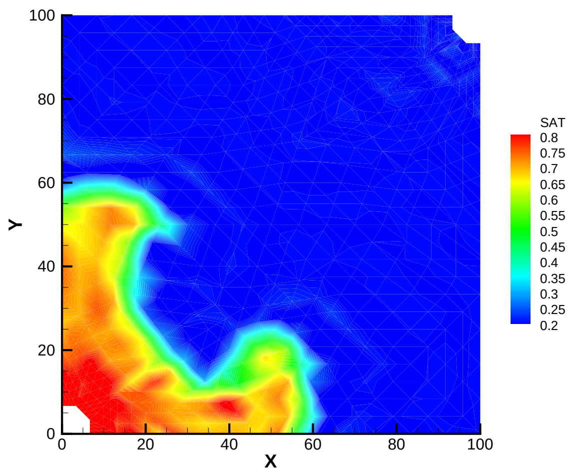


FIGURE 2. (top) Three-dimensional view of pressure contours; (bottom) two-dimensional view of saturation contours at 900 days for heterogeneous benchmark problem: $(r_p, r_s) = (5, 3)$.

In this work, we present stable high-order numerical method based on NIPG in space and backward Euler in time. Due to local approximation, the scheme is well-suited for discontinuous permeability fields and complex geometries. The computational cost is identical to the cost of sequential DG method for two-phase flow. However, the fully implicit approach does not need the use of stabilization techniques such as slope limiters or upwinding. Even in the case of discontinuous permeability fields, the method is stable and robust. As future work, we plan to investigate automatic adaptivity with respect to the mesh and to the polynomial degree to obtain the same accuracy at a lower cost.

References

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