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GLOBAL EXISTENCE FOR THE ONE-DIMENSIONAL SEMILINEAR TRICOMI-TYPE EQUATIONS

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The Tricomi equation

$$u_{tt} - tu_{xx} = 0$$

is a linear partial differential operator of mixed type. (For $t > 0$, the Tricomi equation is hyperbolic; for $t < 0$, it is elliptic; while at $t = 0$, it has multiple characteristics.) In [4] and [5], one can find out about applications of the Tricomi equation. Although this equation is very well investigated, there is no result on the global in time, $t \geq 0$, existence in the Cauchy problem for the nonlinear, even semilinear, Tricomi equation.

It must be emphasized that a one-dimensional wave equation essentially differs from the higher dimensional ones, $x \in \mathbb{R}^n, u_{tt} - \Delta u = 0$. In particular, Theorem 1 shows that there is no self-similar solution to a one-dimensional wave equation, while for a three and higher-dimensional wave equation, existence of such solutions recently is established (Pecher, Ribaud, Youssfi, Ozawa).

Glassey [2] studied the initial value problem,

$$u(x,0) = \psi_0(x), \quad u_t(x,0) = \psi_1(x), \qquad x \in \mathbb{R}^n, \tag{1}$$

for the semilinear wave equation

$$u_{tt} - \triangle u = |u|^{\alpha+1}, \tag{2}$$

with $n = 1, 2, 3$. In particular, for the one-dimensional wave equation,

$$u_{tt} - u_{xx} = |u|^{\alpha+1}, \qquad u(x,0) = \psi_0(x), \quad u_t(x,0) = \psi_1(x), \tag{3}$$

according to the results of [2], there is a finite time blow up for smooth compactly supported initial data if $\alpha > 0$. Namely, if initial data are C^∞ smooth functions and have compact supports and positive averages, then for any $\alpha > 0$, a *classical solution* of (2) cannot exist on $\mathbb{R}^n \times [0, \infty)$.

To motivate our approach consider integral equation

$$u(x,t) = u_0(x,t) + G[|u|^{\alpha+1}](x,t), \tag{4}$$

corresponding to (3), where the function

$$u_0(x,t) = \frac{1}{2}\Big(\psi_0(x+t) + \psi_0(x-t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} \psi_1(y)\, dy$$

is the solution to the Cauchy problem for the linear string equation, while

$$G[f](x,t) \;=\; \frac{1}{2}\int_0^t d\tau \int_{\tau-t}^{t-\tau} f(x+z,\tau)dz\,.$$

If $\psi_0, \psi_1 \in C_0^\infty$, then $u_0 \in C^\infty([0,\infty)\times\mathbb{R})$, and $u_0(t,\cdot)$ obeys the finite speed of propagation property. In particular, it has a compact support for every given instant t. Then, for every fixed $T>0$ operator, G is continuous:

$$G : C([0,T]\times\mathbb{R}) \longrightarrow C([0,T]\times\mathbb{R}) \text{ and}$$

$$G : C([0,T];L^p(\mathbb{R})) \longrightarrow C([0,T];L^q(\mathbb{R})).$$

Any (distributional or classical) solution to the semilinear string equation also solves integral equation (4) with some function $u_0(t,x)$, which is a (distributional or classical) solution to the linear problem. On the other hand, if $u_0 \in C^2([0,T]\times\mathbb{R})$ is given, then any function $u \in C([0,T]\times\mathbb{R})$ which solves integral equation (4) is also solution of the semilinear equation. Denote $K(x_0,t_0) := \{(x,t)\,|\,|x-x_0| < (t_0-t),\ 0<t<t_0\}$. The integral equation (4) is said to be obeying the *Finite Speed of Propagation Property*, if for every point (x_0,t_0) from $u_0(x,0)=0$, $\partial_t u_0(x,0)=0$ on $\{x\in\mathbb{R}\,;\,|x-x_0|\le t_0\}$ and from $u_0=0$ on $K(x_0,t_0)$, it follows $u=0$ on $K(x_0,t_0)$ for the solution $u \in C([0,T];L^q(\mathbb{R}))\cap C^1([0,T];\mathcal{D}'(\mathbb{R}))$.

Theorem 1. *Suppose that $\alpha>0$. For any given nontrivial function u_0 from $C^\infty([0,\infty)\times\mathbb{R})$ generated by $\psi_0,\psi_1 \in C_0^\infty(\mathbb{R})$, $\int_{-\infty}^{\infty}\psi_1(x)dx \ge 0$, there is no global in time solution $u \in C([0,\infty);L^q(\mathbb{R}))\cap C^1([0,T];\mathcal{D}'(\mathbb{R}))$ to the integral equation (4), which obeys the finite speed of propagation property.*

The next conjecture stresses the difference between one-dimensional and higher dimensional semilinear wave equations.

Conjecture (Strauss [7])**.** *For $n\ge 2$ blow-up for all data if $p<p_n$ and global existence for all small data if $p>p_n$.*

Here, $n\ge 2$, $p=\alpha+1$, and p_n is the positive root of the equation $(n-1)p_n^2-(n+1)p_n-2=0$. For the history of the results that have validated Strauss's Conjecture, see [1].

In fact, the above-mentioned difference between $n=1$ and $n\ge 2$ is more importunate. In particular, if we turn to nonsmooth data, for example, homogeneous, like $\psi_0(x)=|x|^{-a}, (a>0)$ and $\psi_1(x)=|x|^{-b}, (b>0)$, then it does not help to avoid a nonexistence. In the case of homogeneous initial data, the solution is self-similar: $u(x,t)=\lambda^{2/\alpha}u(\lambda x,\lambda t)$ for all $\lambda>0$, $x\in\mathbb{R}$, $t>0$. Indeed, if we look for the self-similar solution to the Cauchy problem

$$u_{tt}-u_{xx}=|u(x,t)|^{\alpha+1}\,,\quad u(x,0)=0\,,\quad u_t(x,0)=\varepsilon|x|^{-b}\,,\quad \alpha>0\,,$$

then we have to set $b=1+2/\alpha>1$. On the other hand, the solution $u_0=u_0(x,t)$ of the Cauchy problem for the linear equation,

$$(u_0)_{tt}-(u_0)_{xx}=0\,,\quad u_0(x,0)=0\,,\quad (u_0)_t(x,0)=\varepsilon|x|^{-b}\,,$$

is given by $u_0(x,t) = \int_{x-t}^{x+t} |s|^{-b}/2\, ds$. In this case, since $u(x,t) \geq u_0(x,t)$, the singularity of initial data is spread over the whole light cone and consequently, *nonlinear one-dimensional wave equation does not have self-similar solution.*

Well-posedness of the Cauchy problem for n-dimensional linear Tricomi and Tricomi-type equations was proved by many authors, among them O. A. Oleĭnik [6]. For the progress made in mixed type equations and transonic flow, see C. S. Morawetz [5] and B. L. Keyfitz [3].

In this paper, we establish global existence in the Cauchy problem for the semilinear one-dimensional wave equation with the time dependent coefficient. Namely, consider the Cauchy problem (1) for the following equation,

$$u_{tt} - t^{2k} u_{xx} = \gamma(t) F(u)\,, \tag{5}$$

with $k \geq 0$, $x \in \mathbb{R}$, $t \geq 0$, a continuous function $\gamma \in C((0,\infty))$, while $F(u)$ is

$$F(u) = |u|^{\alpha} u \qquad \text{or} \qquad F(u) = |u|^{\alpha+1}\,, \qquad \alpha > 0\,, \tag{6}$$

$u = u(x,t)$ is real-valued. We assume that for real-valued function γ, we have

$$|\gamma(t)| \leq C t^m \quad \text{for all} \quad t \in (0,\infty)\,,$$

with some constants C and m, $m > -1$.

We study the Cauchy problem (5), (1) through integral equation. To write that integral equation, we appeal to [9, 3.3], where the following operator G is introduced:

$$\begin{aligned} G[f](x,t) \;=\; & (k+1)^{-\frac{k}{k+1}} 2^{-\frac{1}{k+1}} \int_0^t db \int_{x-(\phi(t)-\phi(b))}^{x+\phi(t)-\phi(b)} dy\, f(y,b) \\ & \times (x-y+\phi(t)+\phi(b))^{-\gamma} (\phi(b)-(x-y)+\phi(t))^{-\gamma} \\ & \times F\left(\gamma,\gamma;1; \frac{(x-y+\phi(t)-\phi(b))(x-y-\phi(t)+\phi(b))}{(x-y+\phi(t)+\phi(b))(x-y-\phi(t)-\phi(b))}\right),\ t \geq 0\,. \end{aligned}$$

Here, $F\big(a,b;c;z\big)$ is the hypergeometric function, and

$$\phi(t) := \frac{t^{k+1}}{k+1},\ \gamma := \frac{k}{2}\phi(1).$$

As a matter of fact, operator G is a resolving operator for the Cauchy problem with zero initial data for the linear Tricomi equation,

$$u_{tt} - t^{2k} u_{xx} = f(x,t).$$

Now consider the integral equation

$$u(x,t) = u_0(x,t) + G[\gamma F(u)](x,t)\,, \qquad x \in \mathbb{R}\,,\ t \geq 0\,, \tag{7}$$

where function $u_0 \in C([0,\infty); L^q(\mathbb{R}))$ is given. The following theorem is the main result of the present communication.

Theorem 2. *Assume that*

$$q > 1\,, \quad \beta(\alpha+1) - m < 1\,, \quad \beta = \frac{2+m}{\alpha} - \frac{(k+1)}{q}\,. \tag{8}$$

Let $u_0 \in C([0,\infty); L^q(\mathbb{R}))$ *be given such that*

$$\sup_{t>0} t^\beta \parallel u_0(t) \parallel_{L^q(\mathbb{R})} \leq \varepsilon\,.$$

If ε *is sufficiently small, then there exists a unique solution*

$$u \in C([0,\infty); L^q(\mathbb{R}))$$

of equation (7) *such that*

$$\sup_{t>0} t^\beta \parallel u(t) \parallel_{L^q(\mathbb{R})} \leq 2\varepsilon\,.$$

If a given function $u_0 \in C([0,\infty); L^q(\mathbb{R}))$ solves the Cauchy problem for the linear equation

$$(u_0)_{tt} - t^{2k}(u_0)_{xx} = 0, \qquad u_0(x,0) = \psi_0(x), \quad u_{0\,t}(x,0) = \psi_1(x),$$

then we proved the following global existence theorem for the small data.

Theorem 3. *Assume that conditions* (8)*, and inequalities*

$$\frac{2+m+\alpha}{\alpha(\alpha+1)(k+1)} \leq \frac{1}{q} < \frac{k}{2\alpha(k+1)}\,, \qquad \frac{1}{q} \leq \frac{2+m}{\alpha(k+1)}$$

are satisfied. Let ψ_0*,* ψ_1 *be smooth functions with the compact supports,* ψ_0*,* $\psi_1 \in C_0^\infty(\mathbb{R})$ *and small norms:*

$$\parallel \psi_0 \parallel_{L^p(\mathbb{R})} + \parallel \psi_1 \parallel_{L^p(\mathbb{R})} \leq \varepsilon\,.$$

Here p *submits to terms* $1 < p < \rho'$*,* $1/q = 1/p - 1/\rho'$*,* $1/\rho + 1/\rho' = 1$*.*

If ε *is sufficiently small, then there exists a unique solution*

$$u \in C([0,\infty); L^q(\mathbb{R})) \bigcap C^1([0,\infty); \mathcal{D}'(\mathbb{R}))$$

of the Cauchy problem (1)*,* (5) *such that*

$$\sup_{t>0} t^\beta \parallel u(t) \parallel_{L^q(\mathbb{R})} \leq 2\varepsilon\,.$$

According to the next theorem, the condition $0 < \alpha < (2+m)/k$ is necessary for the existence of the global in time *weak solution* to the problem (1), (5). It generalizes the statement of Theorem 1 to the Tricomi-type equations. In the next theorem, the finite propagation speed property is defined by $K_k(x_0,t_0) := \{(x,t)\,|\,|x-x_0| < (t_0^{k+1} - t^{k+1})/(k+1),\ 0 < t < t_0\}$.

Theorem 4. *Suppose that* $0 < \alpha < (2+m)/k$ *and* $2k \in \mathbb{N}$*. For any given nontrivial function* $u_0 \in C^\infty([0,\infty) \times \mathbb{R})$ *solution to the Cauchy problem for the linear Tricomi-type equation,*

$$(u_0)_{tt} - t^{2k}(u_0)_{xx} = 0\,,$$

with the initial data $\psi_0, \psi_1 \in C_0^\infty(\mathbb{R})$, $\int_{-\infty}^{\infty} \psi_1(x)dx \geq 0$, *there is no global in time solution* $u \in C([0,\infty); L^q(\mathbb{R}))$ *to the integral equation*

$$u(x,t) = u_0(x,t) + G[|u|^{\alpha+1}](x,t),$$

which obeys the finite speed of propagation property.

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