ON THE UNIQUENESS OF THE SUBSONIC DETONATION WAVES IN INERT POROUS MEDIUM

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Gaseous detonation is one of the classical problems of combustion theory. In the past decade, there has been significant progress in understanding the phenomenon. The problem is, however, far from being completely resolved. We study a model of subsonic detonation that describes propagation of the combustion fronts in highly resistible media [5]. The assumption of high resistance of the medium provides a natural simplification of the system of governing equations while still preserving many of the qualitative features of the original system. The model reads:

$$
\begin{aligned}
T_t - (1-\gamma^{-1})P_t &= \varepsilon T_{xx} + Y\Omega(T), \\
P_t - T_t &= P_{xx}, \\
Y_t &= \varepsilon \mathrm{Le}^{-1} Y_{xx} - \gamma Y \Omega(T).
\end{aligned}
\tag{1}
$$

Here P, T, and Y are the appropriately scaled pressure, temperature, and concentration of the deficient reactant, respectively; $\gamma > 1$ is the specific heat ratio, ε is a ratio of molecular and pressure diffusivities, Le is a Lewis number, and $Y\Omega(T)$ is the reaction rate. The first and third equations of the system (1) represent the partially linearized equations for the conservation of energy and deficient reactant, while the second one is a linearized continuity equation taking into account the equations of state and momentum (Darcy's law).

One of the most distinctive features of premixed combustion is its ability to form a reaction wave that assumes the shape of a sharp front propagating subsonically or supersonically at a well defined speed. The front-like solutions of (1), i.e., solutions of the form $T(x,t) = T(\xi)$, $P(x,t) = P(\xi)$, $Y(x,t) = Y(\xi)$ where $\xi = x - ct$ and c is the a priori unknown front speed, satisfy a reduced system of ODE's

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$$\begin{aligned} -cT' + c(1-\gamma^{-1})P' &= \varepsilon T'' + Y\Omega(T), \\ P'' &= c(T'-P'), \\ cY' + \varepsilon Y'' &= \gamma Y \Omega(T) \end{aligned} \tag{2}$$

and the boundary conditions:

$$\begin{aligned} &P(-\infty) = 1, \quad T(-\infty) = 1, \quad Y(-\infty) = 0, \\ &T(+\infty) = 0, \quad P(+\infty) = 0, \quad Y(+\infty) = 1. \end{aligned} \tag{3}$$

Unlike the situation in some other thermo-diffusive systems, the results of this paper do not depend on the value of the Lewis number. For simplicity, we set Le $= 1$. We assume that the function $\Omega(T)$ is of the Arrhenius type with an ignition cut-off, that is, $\Omega(T)$ vanishes on an interval $[0, \Theta]$ and is positive for $T > \Theta$: $\Omega(T) = 0 \quad \text{for} \quad 0 \leq T < \Theta < 1$. Moreover, $\Omega(T)$ is an increasing Lipschitz continuous function, except for a possible discontinuity at the ignition temperature $T = \Theta$. The translational invariance of the system is fixed by assuming that the fronts in consideration reach the ignition temperature Θ at $\xi = 0$.

Most relevant for applications is the case when ε is small. For ε small, one may formally distinguish two separate regimes of propagation: deflagration associated with the small (order of $\sqrt{\varepsilon}$) thermal diffusivity and detonation associated with order one diffusion of pressure [5]. Setting $\varepsilon = 0$, that is ignoring the thermal diffusivity, is very attractive for studying subsonic detonation regime and is believed to reflect the correct phenomenon [5]. For the leading order asymptotics, the system of governing equations (2) reduces to the following one [5].

$$\begin{aligned} -cT' + c(1-\gamma^{-1})P' &= Y\Omega(T), \\ P'' &= c(T'-P'), \\ cY' &= \gamma Y \Omega(T). \end{aligned} \tag{4}$$

On the other hand, there exist arguments supporting the expectation that, in some limiting situations, the details of propagation of the detonation waves are sensitive to the presence of nonzero thermal diffusivity [6].

Solutions of the problem (4) are well understood. In particular, it is known that there is a unique value of $c = c_0$ for which the solution exists [2]. Moreover, as was shown in [3], solutions of (2) converge to that of (4) as $\varepsilon \to 0$. Uniqueness of solutions of the system (2) has not been established before now. It is of interest to see if uniqueness of the wave solution of (4) is robust under perturbation (2) with non-zero ε. We show here that the solution of the system (2) is, indeed, unique for small $\varepsilon > 0$. In Theorem 1, we assume that $\Omega(T)$ is smooth and $\Omega(T) = 0$ for $T < \Theta$.

Theorem 1. *There exist $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$, there is a unique value of c, depending on ε, for which system (2) has an orbit satisfying (3). Moreover, the orbit is unique, and hence, so is the traveling wave up to translation.*

The system (2) is a singular perturbation of (4) as the higher order derivatives are being added in (4). We study the problem using the techniques of geometric singular perturbation theory [4] and show that the traveling wave of (4) perturbs to a unique wave of (2) for $\varepsilon > 0$ small enough.

The strategy of the proof of our main result can be described as follows. We first construct a smooth manifold M_0 on which the traveling wave of the unperturbed system (4) lives. This manifold is normally hyperbolic and therefore, by [1] for small enough $\varepsilon > 0$, perturbs to a unique invariant manifold M_ε of the system (2). With the new variable

$$Q = \frac{1}{c}\int_{\xi}^{+\infty} Y\Omega(T)\,\mathrm{d}x,$$

(2) reads

$$\begin{aligned} Q' &= -c^{-1}Y\Omega(T), \\ P' &= c(T-P), \\ \varepsilon T' &= c(1-\gamma^{-1})P - cT + cQ, \\ \varepsilon Y' &= c(1-Y) - \gamma c Q. \end{aligned} \tag{5}$$

From (5), we obtain the critical manifold M_0

$$(1-\gamma^{-1})P - T + Q = 0, \qquad 1 - Y - \gamma Q = 0, \tag{6}$$

which for $\varepsilon > 0$ but small, at least over compact sets, perturbs to a smooth invariant manifold M_ε. Properties of M_ε allow us to prove that in a neighborhood of M_ε no traveling wave can exist off M_ε. Therefore, we can restrict the flow to M_ε, thus obtaining a lower dimensional problem. On the other hand, using the smooth dependence of M_ε on ε ($0 \le \varepsilon \ll 1$), we extrapolate the information about the existence of a unique front on M_0 to M_ε.

More precisely, we extend the phase space of the system describing the flow on M_0 by adding a direction corresponding to the velocity c:

$$\begin{aligned} Q' &= c^{-1}(\gamma Q - 1)\Omega(Q + (1-\gamma^{-1})P), \\ P' &= -c\gamma^{-1}P + cQ, \\ c' &= 0. \end{aligned} \tag{7}$$

Using properties obtained in [2], we show that the front is represented as a transversal intersection of two invariant manifolds suspended in $M_0 \times \{c \text{ near } c_0\}$. The intersection occurs at a unique value $c = c_0$.

Upon switching on a sufficiently small $\varepsilon > 0$, the transversal intersection perturbs with a nearby c_ε replacing c_0, thus proving the existence of a, unique up to translation, front solution of (2).

The construction of the front using methods of geometric singular perturbation theory also implies that the ε-dependent family of fronts supported by (2) converge to the front of (4) as $\varepsilon \to 0$, thus providing a proof alternative to the one presented in [3].

The full text of the paper is submitted for publication. The preprint ("Traveling waves in porous media combustion: uniqueness of waves for small thermal diffusivity") is available at `http://web.njit.edu/~peterg/PREPRINTS/preprints.html`.

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