FRONT DYNAMICS IN
NON-SMOOTH IGNITION SYSTEMS

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1. Introduction

We consider a non-smooth system which models front motion in a noisy, excitable media. Standard methods from matched asymptotics permit the construction of a family of traveling front solutions. Similar constructions have been carried out in [5] and [2], where travelling wave solutions of non-smooth systems with piecewise linear but discontinuous nonlinearity with fixed jump are constructed for models arising in mechanics and neurobiology.

The model considered here arises from a consideration of membrane hydration in PEM fuel cells in which the protonic transport is approximated as a discontinuous function of membrane water content. Specifically, the membrane is “ignited” if the water content crosses a specific threshold, while it is “extinguished” when water content is below the threshold. We show that the pulse evolution fits naturally into the framework of the renormalization group methods developed to study the stability of slowly evolving patterns [4], [3]. We construct a family of smooth, monotone, composite solutions which describe the propagation of the ignited region across the fuel cell. In a noisy environment, the fronts lose monotonicity and the front position is unresolved because of multiple ignition points. In linearizing about the global manifold of the slowly evolving fronts, we address the challenge of choosing an appropriate Sobolev space for which the nonlinearity is Fréchet differentiable. Our renormalization group approach exploits the fact that the time dependence of the linearized operator is on a slower time scale than the decay. The main result describes the exponential decay of the remainder and after the decay of the initial transient perturbations, we recover the formal pulse velocities at leading order.

2. Model

The following model is a singular perturbation ignition problem with time variable $t$ and spatial variable $y$,

\begin{equation}
U_t = F(U) = \varepsilon U_{yy} - (U - g(y)) + \mu \sigma(U) + \epsilon \eta(y, t).
\end{equation}
The background \( g(y) \) is a smooth, monotone increasing function and the noise function \( \eta \in \mathcal{L}^\infty(\mathbb{R}^+, H^{-\gamma_n}) \), where \( \gamma_n \in (\frac{1}{2}, 1) \). The diffusivity coefficient is asymptotically small \( 0 < \epsilon \ll 1 \), and the hydration conductivity \( \mu \) is constant and satisfies \( 0 < \mu < 1 \). The ignition function \( \sigma \), which models the conductivity of membrane as a function of hydration, is a discontinuous function with a fixed jump \( \sigma_c \) given by:

\[
\sigma(U) = \begin{cases} 
0 & U < \sigma_c \\
U & U > \sigma_c.
\end{cases}
\]

By standard methods of matched asymptotics, we construct a one-parameter family of composite solutions \( \phi(y; y_0, v) \), depicted in Figure 1(a), with an \( \mathcal{O}(\sqrt{\epsilon}) \) front width and a position dependent velocity \( v = v(y) = \mathcal{O}(\sqrt{\epsilon}) \). The front velocity \( v = v(y_0) \) is uniquely determined by the condition that the front \( \phi(y; y_0, v) \) is \( C^1 \) and piecewise \( C^\infty \). To set up the coordinate system for the RG procedure, we fix the slowly evolving front location at time at \( t \), where \( \bar{y}_0 = y_0(t) \), and move the frame with a velocity \( \sqrt{\epsilon} v(\bar{y}_0) \) by the following transformation,

\[
z = y - \bar{y}_0 - \sqrt{\epsilon} v(\bar{y}_0)(t - t_0) / \sqrt{\epsilon}.
\]

For notational convenience, we denote the front location in new coordinate system as \( \bar{z}_0 \), i.e., at \( t = 0, z_0(0) = z_0 = 0 \). Under the transformation (2.2) after rescaling, \( F \) takes the form,

\[
U_\tau = F(U) = U_{zz} + v_s U_z - (U - g_s(z)) + \mu \sigma(C) + \epsilon^{3/4 - \gamma_n/2} \eta_s,
\]

where \( \eta_s \) is scaled so that, \( \|\eta_s\|_{H^\gamma} = \mathcal{O}(1) \), and scaled velocity \( v_s \) and scaled background \( g_s \) are slowly varying functions. The residual \( R = F(\phi) \) in the
convected frame is given by,
\[ R = -(v_s(z_0) - v_s(\bar{z}_0)) \frac{\partial \phi_{\text{inner}}}{\partial z} + \epsilon \left[ \frac{\partial^2 \phi_{\text{outer}}}{\partial z^2} \right]. \]

We decompose the solutions in the neighbourhood of \( \phi \),
\[ U = \phi(z; z_0) + W(z, t). \]

Substituting for \( U \) from (2.4) in the pde (2.3), we wish to Taylor expand \( F(\phi + W) \),
\[ W_t + \frac{\partial \phi}{\partial z} z'_0 = F(\phi) + L\bar{\phi}W + (L\phi - L\bar{\phi})W + N(W) \]
where \( L\bar{\phi} \) is the Fréchet derivative of \( F \) at \( \bar{\phi} = \phi(z; \bar{z}_0) \) in \( H^\gamma \) and \( N \) is the nonlinear operator.

3. The Fréchet Differentiability of Nonlinearity

To justify the expansion on the right-hand side of (2.5), we derive \( L\bar{\phi} : H^\gamma \rightarrow H^{-\beta} \), the Fréchet derivative of \( F \) at \( \bar{\phi} \). We choose \( \gamma \) and \( \beta \) suitably, so that the nonlinearity \( \sigma : H^\gamma \rightarrow H^{-\beta} \) is Fréchet differentiable, but not so large that the semigroup \( S(t) : H^{-\beta} \rightarrow H^\gamma \) fails to be integrable in time.

The Hölder continuity of functions from \( H^\gamma \), for \( \gamma \in (1/2, 3/2) \),
\[ |f(y) - f(x)| \leq C |y - x|^{\gamma - 1/2} \|f\|_{H^\gamma}, \]
is used in the proof of the following lemma, where for notational convenience, we have introduced the tensor product,
\[ (f \otimes g)W = (W, g) f. \]

**Lemma 3.1.** The Fréchet derivative of \( F \) in \( H^\gamma \) for \( \gamma > 1/2 \), at the composite solution \( \bar{\phi} \) is given by
\[ L\bar{\phi} = \partial^2_z + v_s \partial_z - I + \mu \chi_{[\bar{z}_0, \infty]} + \mu \sigma_c \phi'(\bar{z}_0) \delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0}. \]

Moreover, for \( W \in H^\gamma \), \( \gamma > 1/2 \), the nonlinearity \( N(W) = F(\bar{\phi} + W) - F(\bar{\phi}) - L\bar{\phi}W \) satisfies
\[ \|N(W)\|_{H^{-\beta}} \leq c \left( \|W\|^{\beta+1/2}_{H^\gamma} + \|W\|^{\gamma+1/2}_{H^\gamma} \right), \]
for any \( 1/2 < \beta < 1 \).

We illustrate the key steps in the proof of this lemma, highlighting the structure and size of the ignition set which leads to the bound on the nonlinearity. We define the ignition set \( E \), as all zeros of \( \phi + W - \sigma_c \),
\[ E = \{ z | \phi(z) + W(z) = \sigma_c \}. \]
From the Hölder continuity of $H_\gamma$ given by (3.1), we obtain an estimate of the length of $E$,

$$|z_l - z_r| \leq C \left( \|W\|_\gamma \right)^{\frac{1}{2\gamma}}$$

which is smaller than the distance of the frozen front location from the set $E$, as depicted in Figure 2(a).

$$\text{dist}(E, \bar{z}_0) \leq C \|W\|_\gamma.$$  

We define $E_-$ as the set of “false negatives” and $E_+$ as the set of “false positives”:

$$\{ E_- = z | \phi + W \leq \sigma_c \} \bigcap \{ \bar{z}_0 \leq z \leq z_r \},$$

$$\{ E_+ = z | \phi + W \geq \sigma_c \} \bigcap \{ z_l \leq z \leq \bar{z}_0 \}.$$  

To motivate the above sets, observe that for $z > \bar{z}_0$ the composite solution satisfies $\phi \geq \sigma_c$. The set $E_-$ is comprised of the points in $\bar{z}_0 \leq z$ for which $\phi \geq \sigma_c$ but $\phi + W \leq \sigma_c$: that is, $E_-$ contains intervals on which the perturbation $W$ extinguishes the reaction. Similarly, $E_+$ contains the intervals on which the perturbation $W$ induces the reaction. In particular, we have the decomposition,

$$\sigma(\phi + W) = \left( \chi_{E_+} + \chi_{[\bar{z}_0, \infty)} - \chi_{E_-} \right)(\phi + W).$$

We study the case when $E$ is to the left of $\bar{z}_0$, illustrated in Figure 2(b), i.e., $E_-$ is empty and the remaining cases can be treated similarly. We partition $E_+$ as,

$$E_+ = E_+^0 \bigcup [z_l^m, \bar{z}_0].$$

![Figure 2. (a) Structure of set $E$. (b) The graph of $\phi + W$ in the neighbourhood of $\bar{z}_0$ where $E_-$ is empty.](image-url)
We remark that generically the interval \([z_m, z_0]\) is the largest share of \(E_+\). We decompose \(\sigma(\bar{\varphi} + W)\) as
\[
\sigma(\bar{\varphi} + W) = (\bar{\varphi} + W)(\chi_{E_0^0} + \chi_{[z_m, \infty]}).
\]
We note that only the ignition function \(\sigma\) contributes to the nonlinearity which, in this case, reduces to
\[
N(W) = \mu \chi_{E_0^0}(\bar{\varphi}) + \mu \left( \chi_{[z_m, \bar{z}_0]}(\bar{\varphi} + W) - \frac{\partial \phi(\bar{z}_0)}{\partial \phi(\bar{z}_0)} W(\bar{z}_0) \delta_{\bar{z}_0} \right).
\]
We observe that the delta function \(\bar{\varphi}(\bar{z}_0)\) \(\bar{\varphi}'(\bar{z}_0)\) from the linearized operator balances the largest share of \(E_+\) in \(\chi_{[z_m, \bar{z}_0]}\) and the length of \(E_0^0\), given by (3.6), results in the estimate (3.4) for the nonlinearity.

4. The Spectrum and Resolvent

The following lemma characterizes the spectrum of \(L_{\bar{\varphi}}\), where we refer to [1] to characterize the essential spectrum. The principle eigenfunction corresponding to the point spectrum, \(\Lambda_0 \in C^0 \cap H^s\) for \(s < 3/2\), has a prescribed jump in the derivative at \(z = \bar{z}_0\).

**Lemma 4.1.** The essential spectrum of \(L_{\bar{\varphi}}\) is contained in the union of the regions inside or on the curves \(S_{\pm}\) given by
\[
S_+ = \{ k | -k^2 - v(\bar{z}_0)ik - 1 = 0 \},
S_- = \{ k | -k^2 - v(\bar{z}_0)ik - 1 + \mu = 0 \}.
\]
The point spectrum of \(L_{\bar{\varphi}}\) consists of a real, negative \(O(\sqrt{\epsilon})\) eigenvalue with multiplicity one.

To establish estimates on the resolvent operator, we resolve \((L_{\bar{\varphi}} - \lambda) u = F\), for \(F \in H^{-\beta}\), by decomposing
\[
L_{\bar{\varphi}} - \lambda = \mathcal{L} + \alpha \delta_{\bar{z}_0} \otimes \delta_{\bar{z}_0} + \mu \chi_{(\bar{z}_0, \infty)},
\]
where the linearized operator \(\mathcal{L}\) is given by,
\[
\mathcal{L} = \partial_z^2 + v_s(\bar{z}_0) \partial_z - (1 + \lambda).
\]
Further, we set \(u = w + \mathcal{L}^{-1} F\), where \(w\) solves
\[
(L - \lambda) w = -\alpha v_s(\mathcal{L}^{-1} F)(\bar{z}_0) \delta_{\bar{z}_0} - \mu \left( \mathcal{L}^{-1} F \right) \chi_{(\bar{z}_0, \infty)}.
\]
Since \(-L_{\bar{\varphi}}\) is a bounded linear operator, it is sectorial which yields the following semigroup estimate, for \(\lambda \in \mathcal{C}\), the contour \(\mathcal{C}\) given by Figure 1(b) and \(\gamma + \beta < 2\),
\[
\|S(t)F\|_{H^\gamma} \leq \frac{Ce^{-\nu t}}{t^{(\gamma + \beta)/2}} \|F\|_{H^{-\beta}}.
\]
5. Nonlinear Stability

We write the evolution for the remainder \( W \) as,
\[
W_t + \frac{\partial \phi}{\partial z_0} z_0' = R + L_{z_0} W + (L_{z_0} - L_0) W + \mathcal{N}(W) + \epsilon^{3/4 - \gamma_n/2} \eta_s(\xi, t),
\]
(5.1)
\[
W(\xi, 0) = W_0.
\]
(5.2)
where \( W_0 \) is the initial remainder, adapted from Proposition 2.2 of [4]. The term \( \Delta L = L_{z_0} - L_0 \) describes the secular growth implicit in \( z_0 \) sliding away from the frozen front location \( \bar{z}_0 \) and takes the form
\[
\Delta L(W) = (v_s(z_0) - v_s(\bar{z}_0)) \partial_z W + \sigma_c \frac{W(z_0)}{\phi'(z_0)} \delta_{z_0} - \frac{W(\bar{z}_0)}{\phi'(\bar{z}_0)} \delta_{\bar{z}_0} + \mu \chi(z_0, \bar{z}_0) W.
\]
(5.3)
The secularity is the most difficult term in the remainder equation and it satisfies
\[
\| \Delta L(W) \|_{H^{-\gamma}} \leq c |z_0 - \bar{z}_0|^{1/2} \| W \|_{H^\gamma},
\]
(5.4)
which is asymptotically irrelevant. Projecting off the small eigenspace, we observe that the pulse is stationary up to leading order since we have removed the \( O(\sqrt{\epsilon}) \) velocity with the change of variables.
\[
z_0' = \mathcal{O} \left( \epsilon^{3/4 - \gamma_n/2} \| \eta_s \|_{L^\infty(H^{-\gamma_n})} + \| W \|_{H^{\gamma+1/2}}^{\gamma+1/2} \right).
\]
(5.5)
Using the semigroup estimates developed in (4.2), the variation of constants formula applied to (5.1) yields the solution
\[
\| W(t) \|_{H^\gamma} \leq
\| S_{z_0}(\Delta t) W_0 \|_{H^\gamma} + \int_{t_0}^t \| S_{z_0}(t - s) \left( \hat{R} + \Delta L(W) + \mathcal{N}(W) + \epsilon^{3/4 - \gamma_n/2} \eta_s(\xi, t) \right) \|_{H^\gamma} ds.
\]
(5.6)
To revert to the original coordinate system, we introduce the rescaled \( H^\gamma \) norm which uniformly controls the \( L^\infty \) norm,
\[
\| W \|_{H^\gamma} = \left( \int_{\mathbb{R}} |W|^2 + \epsilon^{\gamma-1/2} |D^\gamma W|^2 dy \right)^{1/2}
\]
(5.7)
Hence, we have proved the following theorem for \( \gamma \in \left( \frac{1}{2}, 1 \right) \) and \( \gamma_n > \frac{1}{2} \), satisfying \( \gamma + \gamma_n < 2 \).

**Theorem 5.1.** If the initial data \( U_0 \) can be written as
\[
U(\xi, t) = \phi(y; y_s) + W_0(y, t),
\]
where \( \| W_0 \|_{H^\gamma} \) is sufficiently small, then the solution of the governing equation can be decomposed as
\[
U(y, t) = \phi(y; y_0(t)) + W(y, t),
\]
where 
\[
U(\xi, t) = \phi(y; y_s) + W_0(y, t),
\]
where
\[ \|W\|_{H^y}^2 \leq M \left( e^{-\nu t} \|W_0\|_{H^y}^2 + \epsilon^{3/4 - \gamma n/2} \right), \]
and
\[ y_0'(t) = \sqrt{\epsilon} v(y_0) + O(\epsilon^{5/4 - \gamma n/2}). \]

References


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