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THE INVERSE BOUNDARY VALUE PROBLEM IN ANISOTROPIC MEDIA

BORISLAVA GUTARTS

Inverse problems arise naturally in the physical world around us. The inverse boundary value problem consists of gaining some knowledge about the interior of a body from measurements on the boundary. In medical diagnostics, for example, one is interested in determining the location and/or the size of a tumor inside the body from measurements taken just on the outside. There are also applications in the earth sciences, for example, where one uses measurements taken on the surface in order to locate oil or minerals inside earth.

We are interested in uniqueness results for the inverse problems. To state our inverse boundary value problem, first consider the following boundary value problem

$$\begin{aligned} \nabla \cdot (\gamma(x)\nabla u) &= 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} &= f, \end{aligned} \tag{1}$$

where Ω is some bounded subset of $\mathbb{R}^2$, $\partial\Omega$ is C^∞, $f \in C^\infty(\partial\Omega)$, and $\gamma(x)$ is a positive definite symmetric matrix. If (1) has a unique solution for each f, we can define a Dirichlet-to-Neumann operator

$$\begin{aligned} \Lambda &: \partial\Omega \to \partial\Omega \quad \text{by} \\ \Lambda f &= \frac{\partial u}{\partial n}\Big|_{\partial\Omega}, \end{aligned} \tag{2}$$

where n is the exterior unit normal vector to $\partial\Omega$.

The inverse boundary value problem for (1) consists of determining $\gamma(x)$ from the knowledge of Λ. Notice that this problem is equivalent to the impedance tomography problem, where the goal is to use voltage and current measurements on the boundary in order to find the conductivity of a given material. Here, $\gamma(x)$ represents the electrical conductivity of some material, $u(x)\big|_{\partial\Omega}$ is the voltage, and $\frac{\partial u}{\partial n}\Big|_{\partial\Omega}$ is the current on the boundary. So our

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Dirichlet-to-Neumann map is a set of values of $u(x)\big|_{\partial\Omega}$ and corresponding $\frac{\partial u}{\partial n}\big|_{\partial\Omega}$, and is often called the "voltage-to-current" map.

To answer whether Λ determines $\gamma(x)$ uniquely, which is the question that was posed by Calderon [1], there has been a body of work produced in the last few decades. Kohn and Vogelius [6] proved that Λ determines $\gamma(x)$ and all of its derivatives, but only on the boundary, if $\partial\Omega$ is C^∞. Sylvester and Uhlmann [10] proved for $n \geq 3$ that Λ uniquely determines γ, if $\partial\Omega$ is C^∞ and γ is in $C^\infty(\overline{\Omega})$ (global uniqueness). The 2-dimensional case, however, is the hardest, as unlike in $n \geq 3$, the inverse problem for $n = 2$ is not overdetermined. For $n = 2$, Sylvester and Uhlmann [9] showed uniqueness (up to a change of coordinates) for $\gamma(x)$, when $\log(\det \gamma(x))$ is small in C^3. This result was strengthened by Nachman [7], who showed that for $n = 2$, Λ determines $\gamma(x)$ uniquely (up to a change of coordinates) without the extra assumption used in [9]. Further uniqueness results in two dimensions were obtained by Grinevich and Novikov [3], by Isakov and Nachman [4], and by Isakov and Sun [5].

We consider the anisotropic elliptic equation in $\mathbb{R}^2$. We provide a different proof (the article is in preparation) of Nachman's result [7], that the Dirichlet-to-Neumann map determines the coefficients of the equation uniquely, up to a change of coordinates. We believe that the methods used in our proof (e.g., the absence of the essential points) can be used to solve the inverse boundary value problem for a more general second order operator. We prove the following theorem.

Theorem 1. *Let*

$$L_p(x, -i\partial_x) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(\gamma_p^{ij}\frac{\partial}{\partial x_j}),\ p = 1, 2 \tag{3}$$

with corresponding Λ_p *defined as in (2). If* $\Lambda_1 = \Lambda_2$*, then there exists a diffeomorphism* $y = S(x)$*, such that*

$$\gamma_2(y) = \frac{(J_S(x))^T \gamma_1(x) J_S(x)}{det(J_S(x))}$$

where $J_S(x)$ *is the Jacobian matrix of* $y = S(x)$*. Moreover,*

$$S = I \ \text{on}\ \partial\Omega.$$

In our proof, we use the method previously employed by Sylvester [8] and by Eskin and Ralston [2], which consists of reducing the anisotropic conductivity γ to an isotropic one using isothermal coordinates, then after change of dependent variables getting a Schrödinger equation. We also use work of Nachman [7] and $\overline{\partial}$-equation, but our proofs of some key points are different and new.

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Department of Mathematics; California State University, Los Angeles; Los Angeles, CA 90032 USA

E-mail address: gutarts@calstatela.edu