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A RIEMANN PROBLEM FOR THE ISENTROPIC GAS DYNAMICS EQUATIONS

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We study a Riemann problem for the two-dimensional isentropic gas dynamics equations which models transonic regular reflection. When written in self-similar coordinates, the system changes type from hyperbolic to mixed hyperbolic-elliptic. Using the theory of one-dimensional hyperbolic conservation laws, we formulate a free boundary problem in the subsonic region and we outline the main ideas for proving the existence of a local solution.

1. The statement of the Riemann problem

We consider the system

$$\begin{aligned} &\rho_t + (\rho u)_x + (\rho v)_y = 0, \\ &(\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0, \\ &(\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0, \end{aligned} \tag{1.1}$$

where $(x, y, t) \in \mathbb{R}\times\mathbb{R}\times[0,\infty)$, $\rho : \mathbb{R}\times\mathbb{R}\times[0,\infty) \to (0,\infty)$ stands for density, $u, v : \mathbb{R} \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ are velocities, and $p = p(\rho) : (0,\infty) \to (0,\infty)$ is a known function describing pressure. We assume that $c^2(\rho) := p'(\rho)$ is a positive and increasing function for $\rho > 0$. We consider initial data

$$\overline{U}(x, y, 0) := \begin{cases} \overline{U}_1 = (\rho_1, 0, 0), & -ky < x < ky, \\ \overline{U}_0 = (\rho_0, u_0, 0), & \text{otherwise}, \end{cases} \tag{1.2}$$

where $\rho_0 > \rho_1 > 0$ are arbitrary, $k \in (k_C(\rho_0, \rho_1), k_*(\rho_0, \rho_1))$ is specified in terms of ρ_0 and ρ_1 so that transonic regular reflection takes place (for more details see [14]), and $u_0 = \sqrt{1 + k^2}\sqrt{\frac{[\rho][p]}{\rho_0\rho_1}}$. Here, $[\,\cdot\,]$ denotes the jump between $\overline{U}_0$ and $\overline{U}_1$.

2. The free boundary problem

We write (1.1) in self-similar coordinates, $\xi = x/t$ and $\eta = y/t$, and obtain

$$\begin{aligned} &U\rho_\xi + \rho U_\xi + V\rho_\eta + \rho V_\eta + 2\rho = 0, \\ &UU_\xi + p_\xi/\rho + VU_\eta + U = 0, \\ &UV_\xi + VV_\eta + p_\eta/\rho + V = 0, \end{aligned} \tag{2.1}$$

where we use notation $U := u - \xi$ and $V := v - \eta$. From (2.1), we derive a second order equation for density of the form $Q(\rho, U, V) = 0$, where

$$Q(\rho,U,V) := \big((U^2 - c^2)\rho_\xi + UV\rho_\eta + \rho U\big)_\xi + \big((V^2 - c^2)\rho_\eta + UV\rho_\xi + \rho V\big)_\eta$$
$$+ (UV_\eta - VU_\eta)\rho_\xi + (VU_\xi - UV_\xi)\rho_\eta + 2(U_\xi V_\eta - V_\xi U_\eta)\rho. \quad (2.2)$$

We note that given a constant state $\overline{U}_* = (\rho_*, u_*, v_*)$, equation (2.2) is hyperbolic outside of the sonic circle

$$C_{\overline{U}_*} : (u_* - \xi)^2 + (v_* - \eta)^2 = c^2(\rho_*).$$

Using the theory of one-dimensional systems of hyperbolic conservation laws and the notion of quasi-one-dimensional Riemann problems (see Čanić and Keyfitz [1]), we find a solution in the part of the domain where the system is hyperbolic and we formulate a free boundary problem for the subsonic state and the reflected shock.

With the above choice of u_0, each of the two initial discontinuities $x = \pm ky$ results in a shock and a linear wave in the far field. More precisely, the one dimensional Riemann problem, along the discontinuity $x = ky$, $y \geq 0$, results in the shock $S_1 : x = ky + \sqrt{1+k^2}\sqrt{\frac{\rho_0}{\rho_1}\frac{[p]}{[\rho]}}t$, with the state $\overline{U}_1$ on the left and an intermediate state $\overline{U}_a = \left(\rho_0, u_a = \frac{1}{1+k^2}\sqrt{\frac{[\rho][p]}{\rho_0\rho_1}}, v_a = -ku_a\right)$ on the right, and the linear wave $l_1 : x = ky + u_0 t$ with $\overline{U}_a$ on the left and $\overline{U}_0$ on the right. Similarly, the one-dimensional solution along the discontinuity $x = -ky$, $y \leq 0$, consists of the linear wave $l_2 : x = -ky + u_0 t$ connecting $\overline{U}_0$ to an intermediate state $\overline{U}_b = (\rho_0, u_a, -v_a)$ and the shock $S_2 : x = -ky + \sqrt{1+k^2}\sqrt{\frac{\rho_0}{\rho_1}\frac{[p]}{[\rho]}}t$ connecting $\overline{U}_b$ to $\overline{U}_1$. Further, we find the projected point of intersection of the shocks S_1 and S_2 and denote it by $\Xi_s = (\xi_s, 0) := \left(\sqrt{1+k^2}\sqrt{\frac{\rho_0}{\rho_1}\frac{[p]}{[\rho]}}, 0\right)$ and we consider the quasi-one-dimensional Riemann problem at Ξ_s with state $\overline{U}_b$ on the left and state $\overline{U}_a$ on the right. By our choice of the parameter k, there exist two solutions of this problem and each solution consists of two (reflected) shocks and an intermediate state which we generically denote by $\overline{U}_s = (\rho_s, u_s, v_s)$. Moreover, the point Ξ_s is inside the sonic circle $C_{\overline{U}_s}$ (Figure 1(a)).

As a consequence, the two reflected shocks are transonic and curved, and their position depends, via Rankine-Hugoniot jump conditions, on the unknown nonuniform subsonic state $\overline{U}$ behind the reflected shocks.

In our analysis of the free boundary problem in the subsonic region, we follow the approach first presented in a study of the steady transonic small disturbance equation (TSDE) by Čanić, Keyfitz, and Lieberman [6], and later extended to the study of Riemann problems for the unsteady transonic small disturbance equation (UTSDE) by Čanić, Keyfitz, and Kim [2], [4] and by Jegdić, Keyfitz, and Čanić [13], and the nonlinear wave system (NLWS) by Čanić, Keyfitz, and Kim [5] and Jegdić, Keyfitz, and Čanić [12]. A survey on this approach was given by Keyfitz [15].

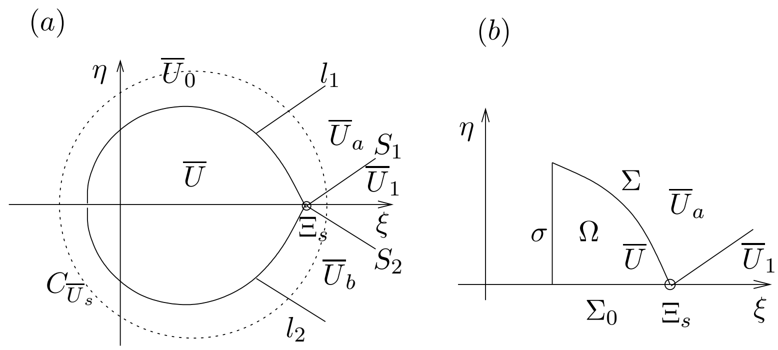

FIGURE 1. The solution in the (ξ, η)-plane and the domain Ω.

Due to symmetry, we consider the problem only in the upper half-plane. We restrict our study to a domain close to the reflection point Ξ_s by introducing a cut-off boundary σ (Figure 1(b)). We impose Dirichlet condition $\rho = f$ along σ, where f is chosen appropriately so that the solution is subsonic along σ (more precisely, $\rho_0 \leq f \leq \rho_s$ and $c^2(f) > m_u^2 + m_v^2$ hold where m_u and m_v are a priori lower bounds for U and V; for more details, see [14]). We impose symmetry conditions for ρ, u, and v along Σ_0, while Rankine-Hugoniot jump relations along Σ are equivalent to (1) an oblique derivative boundary condition $\beta(\rho, U, V) \cdot \nabla \rho = F(\rho, U, V)$ for ρ, (2) Dirichlet conditions for U and V in terms of ρ, and (3) an ordinary differential equation for the reflected shock. Hence, we arrive at the following.

Free boundary value problem: *Let* $\rho_0 > \rho_1 > 0$ *and* $k \in (k_C(\rho_0, \rho_1), k_*(\rho_0, \rho_1))$ *be fixed. Let* $f \in H_{1+\gamma}$, *for* $\gamma \in (0, 1)$ *to be determined, be such that* $\rho_0 \leq f \leq \rho_s$ *and* $c^2(f) > m_u^2 + m_v^2$ *hold. There exists* $\gamma_0 > 0$, *depending on* ρ_0, ρ_1, k *and* $|f|_0$ *such that for every* $\gamma \in (0, \min\{\gamma_0, 1\})$, $\alpha_{\mathcal{K}} \in (0, \gamma)$ *and* $\epsilon \in (0, \alpha_{\mathcal{K}})$, *the problem*

$$\left.\begin{array}{l} Q(\rho, U, V) = 0 \\ (U, V) \cdot \nabla U + U + p_\xi/\rho = 0 \\ (U, V) \cdot \nabla V + V + p_\eta/\rho = 0 \end{array}\right\} \text{ in } \Omega, \tag{2.3}$$

$$\left.\begin{array}{l} \beta(\rho, U, V) \cdot \nabla \rho = F(\rho, U, V) \\ V = V_a + \frac{p(\rho) - p(\rho_0)}{\rho_0(U_a \eta' - V_a)} \\ U = U_a - \eta'(V - V_a) \end{array}\right\} \text{ on } \Sigma : \eta = \eta(\xi), \tag{2.4}$$

$$\frac{d\eta}{d\xi} = \frac{V_a^2 - \overline{c}^2}{U_a V_a + \sqrt{\overline{c}^2(U_a^2 + V_a^2 - \overline{c}^2)}}, \quad \eta(\xi_s) = 0, \tag{2.5}$$

$$\overline{U}(\Xi_s) = \overline{U}_s, \quad \rho_\eta = U_\eta = V = 0 \text{ on } \Sigma_0, \quad \rho = f \text{ on } \sigma, \tag{2.6}$$

has a solution $(\rho, U, V, \eta) \in H^{(-\gamma)}_{2+\epsilon} \times \left(H^{(-\gamma)}_{1+\epsilon}\right)^2 \times H_{1+\alpha_{\mathcal{K}}}$ *in a neighborhood of* Ξ_s.

Here, H denotes appropriate Holder spaces (see [11]), $\overline{c}^2 := \frac{\rho}{\rho_0}\frac{[p]}{[\rho]}$, $U_a := u_a - \xi$, and $V_a := v_a - \eta$. We outline the main ideas of the proof of the above free boundary problem in Ω using the theory of elliptic second order equations with mixed boundary conditions (see Gilbarg and Trudinger [11], Lieberman [16]–[19], and Lieberman and Trudinger [20]) and fixed point theorems. The proof consists of three major steps:

Step 1. We fix Σ within a certain set of admissible curves and we find ρ, U, and V solving the fixed boundary problem (2.3), (2.4), (2.6) in Ω. This proof consists of two parts: we solve the linearized version of the problem and we use a fixed point argument to obtain a solution to the nonlinear problem. Several cut-off functions need to be introduced in order to ensure that the second order equation for ρ is uniformly elliptic, that the condition for ρ on Σ is uniformly oblique, and that the equation (2.5) is well-defined.

Step 2. We use ρ from the previous step and update the position of the free boundary by using the shock evolution equation (2.5). This gives new boundary $\tilde{\Sigma}$, and we show that the map $\Sigma \mapsto \tilde{\Sigma}$ has a fixed point Σ.

Step 3. We take Σ from the previous step and find ρ, U, and V by using Step 1. Finally, we remove the cut-offs introduced in Step 1 in a neighborhood of Ξ_s.

3. Conclusions

Such free boundary problems modeling shock reflection have been considered for various equations (for example, see studies on adiabatic gas dynamics equations by Chang and Chen [7], Euler equations in potential form by Chen and Feldman [8], pressure-gradient system by Zheng [21], steady isentropic gas dynamics equations by Chen [9], and three-dimensional steady Euler system by Chen and Yuan [10]).

The analysis of the above free boundary problem extends previous results on the TSDE, UTSDE, and NLWS. Due to the complicated nonlinear coupling of the isentropic gas dynamics, we were unable to decouple the second order equation and an oblique derivative boundary condition for density from velocities. Moreover, these equations and conditions are inhomogeneous, giving rise to several genuine difficulties.

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