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A LOCALLY DIVERGENCE-FREE NONCONFORMING FINITE ELEMENT METHOD FOR THE REDUCED TIME-HARMONIC MAXWELL EQUATIONS

SUSANNE C. BRENNER, FENGYAN LI *, AND LI-YENG SUNG

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $\boldsymbol{f} \in [L_2(\Omega)]^2$, and $k \geq 0$. Consider the time-harmonic Maxwell equations with the perfectly conducting boundary condition: Find $\boldsymbol{u} \in H_0(\text{curl};\Omega)$ such that

$$(1) \qquad (\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) - k^2(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H_0(\text{curl};\Omega),$$

where

$$H(\text{curl};\Omega) = \Big\{\boldsymbol{v} \in [L_2(\Omega)]^2 : \nabla \times \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \in L_2(\Omega)\Big\},$$
$$H_0(\text{curl};\Omega) = \{\boldsymbol{v} \in H(\text{curl};\Omega) : \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ on } \partial\Omega\}.$$

Here the vector $\boldsymbol{n}$ denotes the unit outer normal on $\partial\Omega$.

Using the Helmholtz/Hodge decomposition [5], we can write

$$\boldsymbol{u} = \mathring{\boldsymbol{u}} + \nabla\phi,$$

where $\mathring{\boldsymbol{u}} \in H_0(\text{curl};\Omega) \cap H(\text{div}^0;\Omega)$ and $\phi \in H_0^1(\Omega)$. Here

$$H(\text{div}^0;\Omega) = \Big\{\boldsymbol{v} \in [L_2(\Omega)]^2 : \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0\Big\},$$

and $\mathring{\boldsymbol{u}}$ and ϕ satisfy the following equations:

$$(2) \quad (\nabla \times \mathring{\boldsymbol{u}}, \nabla \times \boldsymbol{v}) - k^2(\mathring{\boldsymbol{u}}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \text{for all } \boldsymbol{v} \in H_0(\text{curl};\Omega) \cap H(\text{div}^0;\Omega),$$

$$(3) \qquad -k^2(\nabla\phi, \nabla\psi) = (\boldsymbol{f}, \nabla\psi) \quad \text{for all } \psi \in H_0^1(\Omega).$$

In this work, we will focus on (2), which will be referred to as the reduced time-harmonic Maxwell (RTHM) equations. Under the assumption that k^2

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is not a Maxwell eigenvalue, the RTHM equations have a unique solution in $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$.

Our main achievement in this work is that we design a numerical method for RTHM equations using locally divergence-free Crouzeix-Raviart nonconforming P_1 vector fields [4]. And we show that the order of convergence of our method is optimal (up to an arbitrarily small ϵ) in both the energy norm and the L_2 norm, provided properly graded meshes are used. In the spirit of [3], one can say that our results rehabilitate nonconforming nodal finite elements for Maxwell equations. Our method is also the first one designed for solving the divergence-free part of the solution of the Maxwell equation, which will give some new insights for computing Maxwell eigenvalues without introducing spurious eigenmodes. This subject will be addressed in our forthcoming paper.

In the following, we will outline the discrete space on the graded meshes, the numerical scheme, and the main result in error estimate which will be followed by a numerical example. One can refer to [1] for the full details of this work.

2. Locally Divergence-Free Vector Fields on Graded Meshes

Let $\mathcal{T}_h$ be a family of triangulations of Ω. We define the space V_h of locally divergence-free Crouzeix-Raviart nonconforming P_1 vector fields [4] by

$$\begin{aligned} V_h = \{ & \boldsymbol{v} \in [L_2(\Omega)]^2 : \boldsymbol{v}_T = \boldsymbol{v}\big|_T \in [P_1(T)]^2 \text{ and } \nabla \cdot \boldsymbol{v}_T = 0 \ \forall\, T \in \mathcal{T}_h, \\ & \boldsymbol{v} \text{ is continuous at the midpoints of the interior edges of } \mathcal{T}_h, \text{ and} \\ & \boldsymbol{n} \times \boldsymbol{v} = 0 \text{ at the midpoints of the boundary edges.} \end{aligned}$$

In order to recover optimal order convergence for our method, the triangulation $\mathcal{T}_h$ is graded around the corners $c_1, \ldots, c_L$ of Ω and $h_T \approx h\Phi_\mu(T)$ where $h_T = \operatorname{diam} T$, h is the mesh parameter, $\mu = (\mu_1, \ldots, \mu_L)$, and $\Phi_\mu(T) = \Pi_{\ell=1}^{L} |c_\ell - c_T|^{1-\mu_\ell}$. The point c_T is the center of T and μ_ℓ $(1 \le \ell \le L)$ is the grading parameter at the corner c_ℓ. Based on the singularity analysis [2] of the solution of the RTHM equations, we use

$$\begin{aligned} \mu_\ell &= 1 && \text{if} \quad \omega_\ell \le \frac{\pi}{2} \\ \mu_\ell &< \frac{\pi}{2\omega_\ell} && \text{if} \quad \omega_\ell > \frac{\pi}{2}\,, \end{aligned} \tag{4}$$

where ω_l is the interior angle of Ω at c_l.

3. Discretization and an Abstract Error Estimate

Let the set of the interior (resp., boundary) edges of $\mathcal{T}_h$ be denoted by $\mathcal{E}_h^i$ (resp., $\mathcal{E}_h^b$). Let $e \in \mathcal{E}_h^i$ be shared by two triangles $T_1, T_2 \in \mathcal{T}_h$, and let $\boldsymbol{n}_1$ (resp., $\boldsymbol{n}_2$) be the unit normal of e pointing towards the outside of T_1 (resp., T_2). We define, on e,

$$[\![\boldsymbol{n}_e \times \boldsymbol{v}]\!] = \boldsymbol{n}_1 \times \boldsymbol{v}_{T_1}\big|_e + \boldsymbol{n}_2 \times \boldsymbol{v}_{T_2}\big|_e \quad \text{and} \quad [\![\boldsymbol{n}_e \cdot \boldsymbol{v}]\!] = \boldsymbol{n}_1 \cdot \boldsymbol{v}_{T_1}\big|_e + \boldsymbol{n}_2 \cdot \boldsymbol{v}_{T_2}\big|_e .$$

For an edge $e \in \mathcal{E}_h^b$, we take $\boldsymbol{n}_e$ to be the unit normal of e pointing towards the outside of Ω and define $[\![\boldsymbol{n}_e \times \boldsymbol{v}]\!] = \boldsymbol{n}_e \times \boldsymbol{v}\big|_e$.

The discrete problem for the RTHM equations: Find $\mathring{\boldsymbol{u}}_h \in V_h$ such that

$$a_h(\mathring{\boldsymbol{u}}_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \, \text{for all} \, \boldsymbol{v} \in V_h, \tag{5}$$

where

(6)

$$a_h(\boldsymbol{w}, \boldsymbol{v}) = (\nabla_h \times \boldsymbol{w}, \nabla_h \times \boldsymbol{v}) - k^2(\boldsymbol{w}, \boldsymbol{v}) + \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e [\![\boldsymbol{n}_e \times \boldsymbol{w}]\!] \, [\![\boldsymbol{n}_e \times \boldsymbol{v}]\!] \, ds$$
$$+ \sum_{e \in \mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e [\![\boldsymbol{n}_e \cdot \boldsymbol{w}]\!] [\![\boldsymbol{n}_e \cdot \boldsymbol{v}]\!] ds,$$

(7) $\Phi_\mu(e) = \Pi_{\ell=1}^L |c_\ell - m_e|^{1-\mu_\ell}$.

We will measure the discretization error in terms of the norm $\|\cdot\|_h$ defined by

$$\|\boldsymbol{v}\|_h^2 = \|\nabla_h \times \boldsymbol{v}\|_{L_2(\Omega)}^2 + \|\boldsymbol{v}\|_{L_2(\Omega)}^2 + \textstyle\sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|[\![\boldsymbol{n}_e \times \boldsymbol{v}]\!]\|_{L_2(e)}^2$$
$$+ \textstyle\sum_{e \in \mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \|[\![\boldsymbol{n}_e \cdot \boldsymbol{v}]\!]\|_{L_2(e)}^2 .$$

Using the boundness of $a_h(\cdot,\cdot)$ with respect to $\|\cdot\|_h$ and the Gårding's (in)equality, for fixed k, we have the following abstract discretization error estimate under the assumption that (5) is solvable.

Lemma 1. *Let $\mathring{\boldsymbol{u}} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$ satisfy* (2) *and $\mathring{\boldsymbol{u}}_h$ be a solution of* (5)*. It holds that*

(8)

$$\|\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\|_h \le C_k \left(\inf_{\boldsymbol{v} \in V_h} \|\mathring{\boldsymbol{u}} - \boldsymbol{v}\|_h + \max_{\boldsymbol{w} \in V_h \setminus \{\boldsymbol{0}\}} \frac{a_h(\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h, \boldsymbol{w})}{\|\boldsymbol{w}\|_h} + \|\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\|_{L_2(\Omega)} \right).$$

Remark 2. The first term on the right-hand side of (8) measures the approximation property of V_h with respect to the norm $\|\cdot\|_h$. The second term measures the consistency error of the nonconforming discretization. The third term addresses the indefiniteness of the RTHM equations.

4. Convergence Analysis

We collect the estimates for those three terms on the right-hand side of (8) in the following lemma.

Lemma 3. *Let* $\mathring{\boldsymbol{u}} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$ *be the solution of* (2)*, and let* $\mathring{\boldsymbol{u}}_h \in V_h$ *satisfy* (5)*. It holds that*

$$\inf_{\boldsymbol{v}\in V_h} \|\mathring{\boldsymbol{u}} - \boldsymbol{v}\|_h \leq\leq C_\epsilon h^{1-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)} \qquad \textit{for any} \quad \epsilon > 0, \tag{9}$$

$$\max_{\boldsymbol{w}\in V_h\setminus\{\boldsymbol{0}\}} \frac{a_h(\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h, \boldsymbol{w})}{\|\boldsymbol{w}\|_h} \leq Ch\|\boldsymbol{f}\|_{L_2(\Omega)}, \tag{10}$$

$$\|\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\|_{L_2(\Omega)} \leq C_\epsilon\big(h^{2-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)} + h^{1-\epsilon}\|\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\|_h\big) \qquad \textit{for any} \quad \epsilon > 0. \tag{11}$$

Our main result is obtained by following the approach of Schatz for indefinite problems [6].

Theorem 4. *There exists a positive number* h_* *such that the discrete problem* (5) *is uniquely solvable for all* $h \leq h_*$*, in which case the following discretization error estimates are valid*:

$$\|\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\|_h \leq C_\epsilon h^{1-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)} \qquad \textit{for any} \quad \epsilon > 0, \tag{12}$$

$$\|\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\|_{L_2(\Omega)} \leq C_\epsilon h^{2-\epsilon}\|\boldsymbol{f}\|_{L_2(\Omega)} \qquad \textit{for any} \quad \epsilon > 0. \tag{13}$$

We conclude with showing one numerical experiment on the L-shaped domain $(-0.5, 0.5)^2 \setminus [0, 0.5]^2$ to confirm our analytical results (see Table 1). The exact solution is chosen to be

$$\mathring{\boldsymbol{u}} = \nabla \times \Big(r^{2/3}\cos\Big(\frac{2}{3}\theta - \frac{\pi}{3}\Big)\phi(r/0.5)\Big), \tag{14}$$

with $k = 1$, where (r, θ) are the polar coordinates at the origin and the cut-off function is given by

$$\phi(r) = \begin{cases} 1 & r \leq 0.25 \\ -16(r-0.75)^3\big[5 + 15(r-0.75) + 12(r-0.75)^2\big] & 0.25 \leq r \leq 0.75\,. \\ 0 & r \geq 0.75 \end{cases}$$

The meshes are graded around the re-entrant corner with the grading parameter equal to 1/3.

h	$\frac{\Vert\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\Vert_{L_2(\Omega)}}{\Vert\mathring{\boldsymbol{u}}\Vert_{L_2(\Omega)}}$	order	$\frac{\Vert\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\Vert_h}{\Vert\mathring{\boldsymbol{u}}\Vert_h}$	order	$\frac{\vert\mathring{\boldsymbol{u}} - \mathring{\boldsymbol{u}}_h\vert_{\text{curl}}}{\vert\mathring{\boldsymbol{u}}\vert_{\text{curl}}}$	order
1/4	1.23e+02	–	1.88e+01	–	7.77e−00	–
1/8	3.23e+01	1.92	7.38e−00	1.35	4.41e−00	0.82
1/16	2.84e−00	3.50	2.23e−00	1.73	7.87e−01	2.49
1/32	5.84e−01	2.28	1.09e−00	1.03	4.34e−01	0.86
1/64	1.28e−01	2.18	5.50e−01	0.99	2.34e−01	0.90

Table 1. Convergence of the scheme (5) with graded meshes on the L-shaped domain $(-0.5, 0.5)^2 \setminus [0, 0.5]^2$ with $k = 1$ and exact solution $\mathring{\boldsymbol{u}}$ given by (14).

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(Brenner) Department of Mathematics; University of South Carolina; Columbia, SC 29208 USA
E-mail address: brenner@math.sc.edu

(Li) Department of Mathematics; University of South Carolina; Columbia, SC 29208 USA
E-mail address: fli@math.sc.edu

(Sung) Department of Mathematics; University of South Carolina; Columbia, SC 29208 USA
E-mail address: sung@math.sc.edu