A-STABILITY FOR TWO-SPECIES
COMPETITION DIFFUSION SYSTEMS

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We study the structural stability of the attractor ($\mathcal{A}$-stability) for two-species competition diffusion systems

\[
\begin{aligned}
\partial_t u &= k_1 \Delta u + uf(x, u, v), \quad x \in \Omega, \\
\partial_t v &= k_2 \Delta v + vg(x, u, v), \quad x \in \Omega, \\
Bu &= Bv = 0, \quad x \in \partial \Omega,
\end{aligned}
\]

on a $C^2$ bounded domain $\Omega \subset \mathbb{R}^n$ with either Dirichlet or Neumann boundary conditions. Here $u(x, t), v(x, t)$ are the densities of two competing species, $k_1, k_2$ are diffusion constants (called dispersal rates in the ecological literature), and $f, g$ are smooth functions on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ satisfying

(H1) $f(x, 0, 0) > 0, \ g(x, 0, 0) > 0$ for all $x \in \overline{\Omega}$,

(H2) $\partial_u f(\cdot, u, v), \ \partial_v f(\cdot, u, v), \ \partial_u g(\cdot, u, v), \ \partial_v g(\cdot, u, v) < 0$, for all $u, v \geq 0, \ x \in \overline{\Omega}$,

(H3) $\sup_{x \in \overline{\Omega}, v \geq 0} \limsup_{u \to \infty} f(x, u, v) < 0,

(H4) $\sup_{x \in \overline{\Omega}, u \geq 0} \limsup_{v \to \infty} g(x, u, v) < 0$.

These hypotheses describe key features of competition models, and since $u$ and $v$ are the densities of two species, we are only interested in nonnegative solutions $(u, v)$. We, therefore, consider (1) in the positive cone of some appropriate phase space.


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of $\mathcal{A}$-stability (attractor stability), which is more suitable in the infinite-dimensional case, was introduced by J. K. Hale, L. T. Magalhães, and W. M. Oliva, (cf. [2]).

The concept of Morse-Smale structure emerges as a sufficient condition for structural stability. Classically, Morse-Smale refers to systems which have a finite number of critical elements, all of which are hyperbolic and satisfy a transversality condition when their stable and unstable manifolds intersect.

One of the celebrated results due to Palis and Smale, (cf. [7]), states that if a $C^r$-diffeomorphism ($r \geq 1$) on a compact $C^\infty$ manifold without boundary is Morse-Smale, then it is structurally stable. Hale, Magalhães, and Oliva proved that any $f \in KC^r(B,B)$ which is Morse-Smale is $\mathcal{A}$-stable. Here, $B$ is a Banach manifold imbedded in a Banach space $E$ and the choice of the classes $KC^r(B,B)$ depends on the problems under consideration. Another important result in this context is due to K. Lu, (cf. [4]), who proved the "structural stability on a neighborhood of the attractor" for scalar parabolic equations.

Typically, Morse-Smale systems have been defined in the context of (Banach) manifolds with or without boundaries, (cf. [3], [6], etc.), but positive cones do not fall into these categories. Therefore, we first need to modify the classical concepts of Morse-Smale systems and structural stability in such a way that they apply to positive cone settings.

We deal with the openness of Morse-Smale property of (1) and, in the spatially one-dimensional case, we prove that if (1) is a Morse-Smale system on a positive cone, it is $\mathcal{A}$-stable. We also provide a sufficient condition under which the system has the Morse-Smale property. These results will have significant impact on the study of the asymptotic dynamics of various classes of discretizations of (1).

The following are the main results and are stated for the positive cone

$$X_+ \times X_+ := \{ (u, v) \in X \times X | u \geq 0, \ v \geq 0 \text{ on } \Omega \},$$

where $X$ is a suitable fractional power space of

$$-\Delta : \{ u \in H^{2,2}(\Omega) | Bu = 0 \text{ on } \partial\Omega \} \rightarrow L^2(\Omega)$$

satisfying $X \hookrightarrow C^1(\Omega)$.

**Theorem 1.** Let $\mathcal{MS}$ denote for the set

$$\{ (f,g) | f,g \text{ satisfy } (H1)-(H4) \& (1) \text{ has Morse-Smale structure} \}.$$

Then $\mathcal{MS}$ is open.

**Theorem 2.** Assume all the critical elements of (1) are hyperbolic and their union coincides with the non-wandering set $\Omega(f,g)$. Furthermore, suppose that the dimension of the unstable manifold of an equilibrium solution in $X_+ \times X_+ \setminus \{(0,0)\}$ is at most one and the dimension of the unstable manifold of a periodic solution is at most two, then (1) has the Morse-Smale structure.
Theorem 3. Let $\Omega = (0,1)$. Assume (1) is a Morse-Smale system, then it is $A$-stable.

To obtain these results, we view (1) as an evolution equation on $X_+ \times X_+$. Under the conditions imposed on the nonlinearities, we have global existence of solutions with initial values in $X_+ \times X_+$ and existence of a global attractor with respect to solutions in $X_+ \times X_+$.

The proof of Theorem 1 is an adaptation to positive cone setting from an idea used in [5]. For Theorem 2, under the assumptions, we are able to prove the transversality of unstable manifolds and local stable manifolds of critical elements.

The proof of Theorem 3 is quite technically difficult because we have to work with the positive cone $X_+ \times X_+$, not $X \times X$. This proof can be broken down in a few main steps. First of all, since the long-term features of the dynamics of (1) are determined by the global attractor, which lies inside a sufficiently large ball, we can reduce (1) to a finite dimensional system by means of S.-N. Chow, K. Lu, and G. R. Sell’s inertial manifold theorem (cf. [1]). Second, we prove that the finite dimensional system obtained in the first step is also a Morse-Smale system. Third, we prove the $A$-stability of the global attractor (in the positive cone) of the finite dimensional system. The proof adapts the main idea Hale, Magalhães, and Oliva used in [2]. As mentioned before, their result cannot be applied to our problem because we work on a subset of the positive cone and not on a Banach manifold imbedded in a Banach space which is their setting. Because this attractor is the orthogonal projection of the global attractor of (1) (in the positive cone) to the phase space of the finite dimensional system, the final step is to obtain the $A$-stability of the global attractor of (1) (in the positive cone) from the $A$-stability of the global attractor of finite dimensional system.

References


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