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ON THE STABILITY OF STRATIFIED FLOWS WITH A HORIZONTAL SHEAR

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Howard's semi-circle theorem [4] states that the complex wavespeed for an unstable mode must fall within the range of values of the base flow, a conclusion that has proven to be of fundamental importance for a wide variety of flow phenomena. Howard's original proof applies to a two-dimensional basic state, $\overline{u}(z)$, where z is the vertical distance and u is the horizontal velocity, with two dimensional disturbances. Yih [11] demonstrated that two-dimensional disturbances are the most unstable, making the three-dimensional problem unnecessary for $\overline{u}(z)$.

Howard's theorem was extended by Kochar and Jain [7], who proved that the complex wave velocity for any unstable mode lies within a semi-ellipse whose major axis coincides with the diameter of Howard's semi-circle, while its minor axis depends on the stratification. Banerjee, Gupta, and Subbiah [1] found a more limiting version of Howard's theorem by restricting attention to homogeneous flow. Pedloski [8], [9] proved Howard's theorem for flow with a base rotation, which was later improved by Kanwar and Sinha [6]. Dahlburg, Boncinelli, and Einaudi treated the magnetohydrodynamics case [3].

Another basic state of importance in geophysical flow is the case where the base flow varies horizontally, $\overline{u}(y)$, where y is a horizontal distance. This mean flow was considered by Ivanov and Morozov [5], who used a numerical approach to study linear waves with a specific mean flow profile, and Basovich and Tsimring [2], who used the WKB method to also study linear waves specific mean flow profiles. Staquet and Sommeria [10] review this work briefly.

Howard's theorem is extended here to include the case of a flow with vertical stratification, $\overline{\rho}(z)$, and a horizontally varying mean flow, $\overline{u}(y)$. Note that such a flow is inherently three-dimensional, and Yih's theory [11] no longer applies. Furthermore, Rayleigh's theorem is extended to include the case with horizontally varying mean flow, and then Howard's theory is further extended to include the mean flow, varying horizontally and vertically, $\overline{u}(y,z)$, but only for the Boussinesq case.

With stratification, two types of disturbance modes are expected to be present: 1) modes associated with the parallel flow that would exist in a similar form without stratification, and 2) internal waves that would exist without the mean flow, but are distorted by presence of the mean flow. The above theorems of course apply to both types of modes. One may thus conclude that internal waves moving faster than the mean flow cannot be linearly unstable. Internal waves that do travel with the range of the mean flow may be unstable and could grow into large amplitude waves. This potential instability suggests that internal waves with speeds within the range of the mean flow are more likely to be observed in nature.

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