1991 ALGEBRA PRELIMINARY EXAMINATION

The exam is divided into four sections: Group Theory, Ring Theory, Galois Theory and Homological Algebra. In each section, do any three of the four problems.

Group Theory

1. State and prove Cauchy's Theorem.

2. Show that a group is finite if and only if it has finitely many subgroups.

3. If K is a normal subgroup of a finite group G and S is a p-Sylow subgroup of K, prove that $G = \mathrm{N}_G(S)K$.

4. Let G be a group.
   (a) If N is a normal subgroup of G, show that G/N is abelian if and only if N contains the commutator subgroup of G.
   (a) Define the n^{th}-center of G and show that it is a characteristic subgroup of G.

Ring Theory

1. State the Artin-Wedderburn Theorem.

2. Show that every PID is a UFD.

3. Prove or disprove: If R is a PID, the same holds for the polynomial ring $R[x]$.

4. Let R be commutative. Show that

$$\{a \in R : a^n = 0 \text{ for some positive integer } n\ \}$$

   is an ideal of R.

Galois Theory

1. State the Fundamental Theorem of Galois Theory.

2. Compute the Galois group of $f(x) = x^3 - 7$ over $\mathbf{Q}$, and determine a splitting field for f over $\mathbf{Q}$.

3. Show that a field F contains a subfield isomorphic to $\mathbf{Q}$ or $\mathbf{Z}/p\mathbf{Z}$ for some prime p.

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4. Show that a finite field extension is algebraic.

Homological Algebra

1. Show that a ring R is right hereditary if and only if the class of projective right R-modules is closed with respect to submodules.

2. Prove that a right R-module M is flat if all its finitely generated submodules are flat.

3. Let R be a right Artinian local ring. Show that 0 and 1 are the only idempotents of R.

4. Let G be an abelian group. Show that there exists an exact sequence

$$0 \longrightarrow tG \longrightarrow G \longrightarrow \mathbf{Q} \otimes_{\mathbf{Z}} G \longrightarrow T \longrightarrow 0$$

where T is a torsion group.