1997 ALGEBRA PRELIMINARY EXAMINATION

I. Group Theory

In this section of the exam, work one of the first two problems and any three of the remaining five.

1. State the Fundamental Theorem for Finitely Generated Abelian Groups, and give an outline of its proof.

2. State the Sylow Theorems, and prove the First Sylow Theorem.

3. Show that, up to isomorphism, there are exactly two groups of order 10.

4. Let G be a group and suppose N is a normal subgroup of G such that $G/N \cong \mathbf{Z}$, the infinite cyclic group. Show that there is a subgroup U of G such that $U \cap N = \{e\}$ and $G = NU$.

5. Let $\mathbf{Z}_m$ and $\mathbf{Z}_n$ be cyclic groups of orders m and n, respectively. Show that $\mathbf{Z}_m \times \mathbf{Z}_n$ is cyclic if and only if $(m, n) = 1$.

6. Show that no group of order 56 is simple.

7. Show that a group cannot be the union of two proper subgroups.

II. Ring Theory

Work any four of the following five problems. Throughout this section, R is a commutative ring with identity $1 \neq 0$.

1. Let d be a square-free integer and let $R = \mathbf{Z}[\sqrt{d}]$. Argue that for any nonzero ideal I of R, R/I is finite. Conclude that R is a noetherian integral domain of Krull dimension 1.

2. Let R be a noetherian integral domain. Show that every nonzero nonunit of R can be factored into a product of irreducible elements.

3. Let R be a Dedekind domain and let I be a nonzero ideal of R. Show that I is primary if and only if $I = P^m$ for some maximal ideal P of R and positive integer m.

4. Let P_1 and P_2 be distinct maximal ideals of R.
   (a) Show that $P_1^m + P_2^n = R$ for all positive integers m and n.
   (b) Suppose R is noetherian and let I_1 and I_2 be ideals of R such that I_j is P_j-primary. Show that $I_1 + I_2 = R$.

5. Let R be a noetherian local ring with maximal ideal P and let I be a nonzero ideal of R. Use Nakayama's Lemma to prove the following two statements.

   (a) $PI \neq I$.

   (b) If I is invertible, any $r \in I \setminus PI$ generates I. (Consider $rI^{-1} + P$.)