Algebra Prelim, Spring 2004

May 13, 2004

Do **eight** of the following twelve problems. You must show the logical steps in deriving your answers.

1. State the Sylow Theorems.

2. Show that there does not exist a simple group of order p^2q.

3. State the Fundamental Theorem for Finitely Generated Modules Over Principal Ideal Domains.

4. How many Abelian groups are there of order 1800?

5. Show that the following are equivalent for a finite group G:

   i) G is solvable.

   ii) Every non-trivial epimorphic image of G contains a non-trivial normal Abelian subgroup.

   iii) G has a composition series with cyclic factors.

6. Give an outline of the proof that there are five non-isomorphic groups of order 8.

7. Let R be a domain with the ascending chain condition on principal ideals (i.e., if $(a_1) \subseteq (a_2) \subseteq (a_3) \cdots$, then for some $n \geq 1$, $(a_k) = (a_n)$ for all $k \geq n$). Show that every non-zero non-unit element of R is a product of irreducible elements.

8. a) Define Dedekind domain (as done in Algebra II) in terms of prime ideals in the domain and factorizations of ideals.

   b) Give an outline of the proof that every proper ideal in a Dedekind domain is invertible.

9. Give examples of the following:

   a) A Dedekind domain which is not a principal ideal domain.

   b) A UFD which is not Dedekind. (Hint: consider a certain polynomial ring)

10. State the Fundamental Theorem of Galois Theory.

11. Find the Galois group of $f(x) = x^4 - 2$ over the rational numbers.

12. Let E be a splitting field of a separable polynomial over F, and let G be the Galois group of E. Assuming the information that

$$|H| = [E : E_H] \text{ for } H \leq G \text{ where } E_H = \{\alpha \in E \mid \sigma(\alpha) = \alpha \text{ for all } \sigma \in H\},$$

and that

$$|G_K| = [E : K] \text{ for } K, \text{ with } F \subseteq K \subseteq E, \text{ and } G_K = \{\sigma \in G \mid \sigma(\beta) = \beta \; \forall \, \beta \in K\} :$$

Deduce the correspondence part of the Fundamental Theorem of Galois Theory.