March 1992 General Examination in Analysis (administered by J. B. Brown).

Work at least 8 problems.

1.a) Define what it means to say that a subset M of [0,1] is (i) nowhere dense, (ii) first category, (iii) of Lebesgue measure zero, (iv) Lebesgue measurable.

b) Give an example (include details of construction) of a nowhere dense set which is of positive measure.

2. Describe a non-measurable subset of [0,1] and explain why you know it is nonmeasurable.

(Hypothesis for 6-9) Let $f, f_1, f_2, \ldots$ be real valued functions which are measurable with respect to a σ-algebra A on a set Ω, and let μ be a (finite) measure on μ.

3. Define what it means to say that (a) $\{f_n\}$ converges to f in measure μ, (b) $\{f_n\}$ converges to f uniformly, (c) $\{f_n\}$ converges to f almost everywhere (μ), (d) $\{f_n\}$ converges to f in the $L^1(\mu)$ sense, (e) $\{f_n\}$ converges to f pointwise.

4. Line up the notions of convergence of # 3 in-so-far-as which implies which. Give an example which shows that at least two of these implications don't hold if the measure μ is σ-finite rather than finite.

5. Prove that if $\{f_n\}$ converges to f in measure μ, then some subsequence of $\{f_n\}$ converges to f almost everywhere (μ).

6. Prove Egorov's theorem, i.e. that if $\{f_n\}$ converges almost everywhere (μ) to f and $\varepsilon > 0$, then there is a set M such that $\mu(\mathrm{M}^c) < \varepsilon$ and $\{f_n|\mathrm{M}\}$ converges to $f|\mathrm{M}$ uniformly.

7. State the "Lebesgue Dominated Convergence Theorem" (about moving "$\lim_{n\to\infty}$" inside or outside the integral sign).

8. Give two equivalent definitions (an "$\varepsilon-\delta-$partition" definition and another involving Lebesgue integrals) for what it means to say that a function $f\colon [0,1] \to \mathbb{R}$ is absolutely continuous. Give an example of a function $f\colon [0,1] \to \mathbb{R}$ which is continuous and of bounded variation but is not absolutely continuous.

9. Define $\ell^p, L^p[0,1], L^p(\mathbb{R})$, and $L^p(\mu)$ for $0 < p \le \infty$ { you can make the L^p-spaces collections of functions or collections of equivalence classes of functions, either way is OK }.

10. (a) Give an example which shows that $L^p[0,1]$ is not closed under taking of products if $0 < p < \infty$.

    (b) Explain why it is true that if $f \in L^1[0,1]$, then $\sqrt{|f|} \in L^1[0,1]$, but that the same does not hold for $L^1(\mathbb{R})$.

11. Define "Banach Space". For which $0 < p \leq \infty$ is $L^p[0,1]$ $\{$ assuming the equivalence class definition $\}$ a Banach space? What goes wrong for the other p's? What is the norm for the case where $L^p[0,1]$ is a Banach space?