AN ALGEBRAIC CHARACTERIZATION
OF THE FREUDENTHAL
COMPACTIFICATION FOR A CLASS OF
RIMCOMPACT SPACES

by

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1. Introduction

Throughout $C(X)$ will denote the ring of all continuous real-valued functions on a Tychonoff space $X$, and $C^*(X)$ will denote the subring of bounded elements of $C(X)$. The real line is denoted by $\mathbb{R}$, and $\mathbb{N}$ denotes the (discrete) subspace of positive integers. A subset $S$ of $X$ such that the map $f \mapsto f|_S$ is an epimorphism of $C(X)$ (resp. $C^*(X)$) is said to be $C$-embedded (resp. $C^*$-embedded) in $X$. As is well-known, every $f \in C^*(X)$ has a unique continuous extension $\beta f$ over its Stone-Cech compactification $\beta X$ [GJ, Chapter 6]. That is, $X$ is $C^*$-embedded in $\beta X$.

In [NR], L. Nel and D. Riordan introduced the subset $C^\#(X)$ of $C(X)$ consisting of all $f$ such that for every maximal ideal $M$ of $C(X)$, there is an $r \in \mathbb{R}$ such that $(f-r) \in M$, and they noted that $C^\#(X)$ is a subalgebra and sublattice of $C(X)$ containing the constant functions. They show how $C^\#(X)$ determines a compactification of $X$ in a number of cases and leave the impression that it always does. In [Cl], E. Choo notes that this is true if $X$ is locally compact and seems to conjecture that it need not be the case otherwise. In [SZ 1], O. Stefani and A. Zanardo show that every $f \in C^\#(\mathbb{R}^\omega)$ is a constant function, where $\mathbb{R}^\omega$ denotes a countably infinite product of copies of $\mathbb{R}$. In [SZ 2] they show that $C^\#(X)$
determines a compactification of X in case X is locally compact, pseudo compact, or zero-dimensional, and they describe the compactifications so determined when X is real-compact [GJ, Chapter 8].

In this paper, I show that under certain restrictions on X, the ring $C^\#(X)$ determines the Freudenthal compactification of X [11, pp. 109-120], I observe that, at least in disguised form, $C^\#(X)$ has been considered by a number of authors other than those named above, and some conditions are given that are either necessary or sufficient for X to determine a compactification of X. In particular, it is shown that if X is realcompact, and $C^\#(X)$ determines a compactification of X, then X is rimcompact and it determines the Freudenthal compactification $\phi X$ of X. There are realcompact rimcompact spaces X for which $C^\#(X)$ does not determine a compactification of X, but $C^\#(X)$ does determine $\phi X$ if every point of x has either a compact neighborhood, or a base of open and closed neighborhoods. Other sufficient conditions are given for $C^\#(X)$ to determine $\phi X$. I close with some remarks and open problems.

2. Using $C^\#(X)$ to Compactify X

We will make use of the following characterization of $C^\#(X)$ due to a number of authors. Recall that $Z(f) = \{x \in X: f(x) = 0\}$ and $\nu X$ denotes the Hewitt real compactification of X.

2.1 Theorem. If $f \in C(X)$, then the following are equivalent.

(a) $f \in C^\#(X)$. 

(b) \( f \in C^*(X) \) and \( f[D] \) is closed (and hence finite) for every \( C \)-embedded copy \( D \) of \( N \).

(c) \( f \in C^*(X) \) and \( f[Z] \) is closed for every zero-set \( Z \) in \( X \).

(d) \( f \in C^*(X) \) and for every \( r \in R, \text{Cl}_{\beta X} Z(f-r) = Z(\beta f-r) \).

(e) \( f \in C^*(X) \) and for every \( p \in \beta X \setminus \nu X \), there is a neighborhood of \( p \) in \( \beta X \) on which \( \beta f \) is constant.

The equivalence of (a) and (b) seems to appear first in [NR]. The equivalence of (a), (b), (c), (d) appears in [Cl], and that of (a), (b), (d), and (e) in [SZ 2]. Mappings that satisfy (d) are a special case of what are called WZ-maps by T. Isiwata, who showed that any map that sends zero-sets to closed sets in a WZ-map, and that a WZ-map on a normal space is closed [I 2], [W, p. 215]. More important for this paper is the following result. For any subset \( S \) of \( X \), let \( Fr S = Cl S \cap Cl(X \setminus S) \) denote the boundary (or frontier) of \( S \).

2.2 Theorem. If \( X \) is realcompact and \( f \in C^!(X) \), then \( Fr Z(f-r) \) is compact for every \( r \in R \), and \( f \) is a closed mapping.

By Theorem 2.1 (d,e) if \( r \in R \), then either \( Z(f-r) \) is compact or \( Fr Z(\beta f-r) \subset X \). In the latter case, \( Fr Z(f-r) = Fr Z(\beta f-r) \). In either case \( Fr Z(f-r) \) is compact. In [I.2, 1.3], T. Isiwata shows that a WZ-map with this latter property is closed, so the theorem is proved.

Recall that a space \( X \) is called \textit{rimcompact} if it has a base of open sets with compact boundaries. \( X \) is said to be \textit{zero-dimensional} at \( x \) if \( x \) has a base of neighborhoods with
empty boundaries, and $X$ is called zero-dimensional if it is zero-dimensional at each of its points. It is shown in [M3] that every rimcompact space has a compactification $\beta X$ such that $\beta X \setminus X$ is zero-dimensional, and wherever $\gamma X$ is a compactification of $X$ with $\gamma X \setminus X$ zero-dimensional, there is a continuous map of $\beta X$ onto $\gamma X$ leaving $X$ pointwise fixed. $\beta X$ is called the Freudenthal compactification of $X$.

In [D], R. Dickman shows that if $X$ is rimcompact, then every $f \in C^*(X)$ such that $\text{Fr} Z(f-r)$ is compact for every $r \in \mathbb{R}$ has a (unique) extension in $C(\beta X)$. Hence the following is an immediate consequence of Theorem 2.2.

2.3 **Corollary.** If $X$ is rimcompact and realcompact, then every $f \in C^\#(X)$ has a (unique) extension $f^* \in C(\beta X)$.

Suppose $S$ is a subring of $C^*(X)$ that contains the constant functions and $\gamma X$ is a compactification of $X$ such that every $f \in S$ has an extension $\gamma f \in C(\gamma X)$ and $S^\gamma = \{\gamma f : f \in S\}$ separates the points of $\gamma X$. (That is if $x_1', x_2' \in \gamma X$ and $x_1' \neq x_2'$, there is an $f \in S$ such that $\gamma f(x_1') = 0$ and $\gamma f(x_2') = 1$). Then by the Stone-Weierstrass Theorem, $S^\gamma$ is dense in $C(\gamma X)$ in its uniform topology [GJ, 16.4], and we say that $S$ determines the compactification $\gamma X$ of $X$. Note that $S$ determines a compactification of $X$ if points can be separated from disjoint closed sets by functions in $S$.

If $\gamma_1 X$ and $\gamma_2 X$ are compactifications of $X$ for which there is a homeomorphism of $\gamma_1 X$ onto $\gamma_2 X$ keeping $X$ pointwise fixed, then we write $\gamma_1 X = \gamma_2 X$.

For any space $X$, let $C^\#(\beta X) = \{\beta f : f \in C^\#(X)\}$ and note that $C^\#(\beta X)$ and $C^\#(X)$ are isomorphic. Similarly, if $X$ is
realcompact and rimcompact, then by Corollary 2.3, $C^#(X)$ is isomorphic to $C^#(\phi X) = \{ \phi f : f \in C^#(X) \}$.

A subring $A$ of $C^*(X)$ is called algebraic if it contains the constant functions and those members $f \in C^*(X)$ such that $f^2 \in A$. If, in addition, $A$ is closed under uniform convergence, then $A$ is called an analytic subring of $C^*(X)$. The closure in the uniform topology of a subset $B$ of $C^*(X)$ will be denoted by $uB$. It is noted in [GJ, 16.29], that if $A$ is an algebraic subring of $C^*(X)$, then $uA$ is an analytic subring.

If $B \subset C^*(X)$, then a maximal stationary set $S$ of $B$ is a subset of $X$ maximal with respect to the property that every $f \in B$ is constant on $S$. In [GJ, 16.29-16.32], the following is established.

2.4 If $X$ is compact and $A$ is an algebraic subring of $C^*(X)$, then every maximal stationary set of $A$ is connected and $uA = \{ f \in A : f$ is constant on every connected stationary set of $A \}$.

If $X$ is rimcompact and realcompact, then, by the above $C^#(\phi X)$ is an algebraic subring of $C^*(\phi X)$. Next, I make use of the above to establish:

2.5 Theorem. If $X$ is a realcompact space and $C^#(X)$ determines a compactification $\gamma X$ of $X$, then $X$ is rimcompact and $\gamma X = \phi X$.

Proof. Suppose $x \in X$ and $V$ is an open neighborhood of $x$. By assumption there is an $f \in C^#(X)$ such that $f(x) = 0$ and $f(X\setminus V) = 1$. If $g = (f - \frac{1}{2}) \vee 0$, then, by Theorem 2.2 $Z(g)$ is a neighborhood of $x$ with compact boundary that is
contained in $V$. Hence $X$ is rimcompact, and so $A = C_\#(\phi X)$ is an algebraic subring of $C^*(\phi X)$. Assume without loss of generality that $X$ is not compact, let $S$ denote a maximal stationary set of $A$, and suppose $S$ has more than one point. Since $A$ determines a compactification of $X$, it follows that $S \subseteq \phi X \setminus X$. Since the remainder of $X$ in $\phi X$ is totally disconnected, $S$ reduces to a point and Theorem 2.5 is established.

Next, I give an example to show that $C_\#(X)$ need not determine a compactification of a realcompact and rimcompact space. For any space $X$, let $R(X)$ denote the set of points of $X$ which fail to have a compact neighborhood. Clearly $R(X)$ is closed since $X \setminus R(X)$ is open.

2.6 Example. A realcompact rimcompact space $S$ for which $R(X)$ is a compact connected maximal stationary set.

Let $W^*$ denote the space of ordinals that do not exceed the first uncountable ordinal $\omega_1$, and let $W = W^* \setminus \{\omega_1\}$. It is well known that $W^*$ is compact and every $f \in C(W)$ is eventually constant [GJ, 5.13]. Let $X = [0,1] \times W^*$ with the topology obtained by adding to the product topology every subset of $[0,1] \times W$. Clearly $X$ is rimcompact and $R(X) = [0,1] \times \{\omega_1\}$. Moreover, $X$ is the union of a realcompact discrete space and the compact space $R(X)$, so $X$ is realcompact [GJ, 8.16]. Suppose $0 \leq r < s \leq 1$ and $g \in C^*(X)$ is such that $g(r,\omega) \neq g(s,\omega)$. Since $[0,1]$ is connected, since every $f \in C(W)$ is eventually constant, and since $W$ has no countable cofinal subset, there is an $\alpha > \omega_1$, and an increasing sequence $\{x_n\}$ of real numbers between $r$ and $s$ such that $g(x_n,\alpha) \neq g(x_m,\alpha)$ if $n \neq m$. Thus $g$ assumes infinitely many
values on a closed discrete subspace of $X$ and hence cannot be in $C^\#(X)$ by Theorem 2.1(b). So $R(X)$ is a maximal stationary set of $C^\#(X)$.

It is clear that $C^\#(X)$ always contains both the subring $C_K(X)$ of all functions with compact support and the subring $C_F(X)$ of functions with finite range. Clearly any point of $X \setminus R(X)$ can be separated from any disjoint closed set by some element of $C_K(X)$, and if $X$ is zero-dimensional at a point $x$, then $x$ can be separated from any disjoint closed set by some element of $C_F(X)$. This together with 2.4 and Theorem 2.5 proves:

2.7 Theorem. If $X$ is a rimcompact, realcompact space that is zero-dimensional at each point of $R(X)$, then $C^\#(X)$ determines $\phi_X$; that is, $u C^\#(\phi_X) = C(\phi_X)$.

Along these lines we have also:

2.8 Theorem. If $X$ is a rimcompact and realcompact space such that $\text{cl}_{\phi_X}(\phi_X \setminus X)$ is zero-dimensional, then $u C^\#(\phi_X) = C(\phi_X)$.

Proof. By the remarks proceeding the proof of Theorem 2.7, if $S$ is a maximal stationary set for $C^\#(\phi_X)$ with more than one point, then $S \cap \text{cl}_{\phi_X}(\phi_X \setminus X)$. Since the latter set is zero-dimensional, $S$ reduces to a point and the conclusion follows.

In [Il, Theorem 36, p. 114], it is shown that if $\phi_X \setminus X$ is a Lindelöf space, then the Lebesgue dimension of $\phi_X \setminus X$ is zero. In [P, Corollary 5.8] it is shown that if $F$ is a closed subset of a normal space $Y$, then the Lebesgue dimension
of $Y$ does not exceed the Lebesgue dimensions of $A$ or $(Y \setminus A)$. It follows that if $R(X)$ is compact and zero-dimensional, then \( \overline{c_X}(\phi X \setminus X) = (\phi X \setminus X) \cup R(X) \) is zero-dimensional, for these two motions of dimensionality coincide at 0 if $X$ is compact; see [P, pp. 156-157]. Note also that $\phi X \setminus X$ is a Lindelöf space if and only if every compact subset of $X$ is contained in a compact subset with a countable base of neighborhoods; in which case we will say that $X$ is of countable type. [II, p. 119]. Thus we have established:

2.8 Corollary. If $X$ is a rimcompact, realcompact space of countable type, and $R(X)$ is compact and zero-dimensional, then $u_C(\phi X) = C(\phi X)$.

3. Remarks and Open Problems

A. In [N], the ring of all closed $f \in C(X)$ is considered for $X$ locally compact and weakly paracompact ( = metacompact). For $X$ realcompact this latter ring coincides with $C^*(X)$ by Theorem 2.2. Recall also that W. Moran showed in [M3] that if every closed discrete subspace of a normal metacompact space $X$ is realcompact, then so is $X$. Also, examination of Example 3 of [N] shows that this latter need not hold if $X$ fails to be normal.

B. In a private communication S. Willard notes that if $f \in C^*(X)$ and $f$ is a closed mapping, then $Z(f)$ has a countable base of neighborhoods in $X$. (I.e., $Z(f) = \cap_{i=1}^\infty f^{-1}(-1/i, 1/i)$). It would be of great interest to characterize the zero-sets of elements of $C^*(X)$ at least in case $X$ is rimcompact and realcompact. To determine which such spaces determine $X$, it would probably be
enough to characterize zero-sets of restrictions to $X$ of $u \in C_\#(\Phi X)$.

C. Willard notes also that if $S$ is a countable subset of $X$ and $\text{cl}_{\Phi X} S$ is connected, then $S$ is a stationary set for $C_\#(X)$. It follows from a theorem of McCartney [M1, Proposition 3.12] that if $Y = [0,1] \times (0,1] \cup Z$, where $Z = \{(q,0): 0 \leq q \leq 1 \text{ and } q \text{ is rational}\}$, then $\Phi Y = [0,1] \times [0,1]$. Hence, by the latter remark of Willard cited above, $Z$ is a stationary set for $C_\#(Y)$, so $Y$ is a separable, metrizable rimcompact space such that $C_\#(Y)$ does not determine a compactification of $Y$.

D. Suppose $X = [0,1] \times Q \cap [0,1]$, where the open sets of $X$ and those in the product topology together with any subset of $\{(a,b) \in X: b > 0\}$. Then $R(X) = \{(a,b) \in X: b = 0\}$ is compact and connected, $X$ is rimcompact, realcompact, and determines $\Phi X$. So the hypotheses of Theorem 2.7 or 2.8 are not necessary for $X$ to determine $\Phi X$.

References


[SZ2] ________, Alcune caratterizzazioni di una sottoalgebra di $C^*(X)$ e compattificazioni ad essa associate, ibid. 53 (1975), 363-367.


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