SOME REMARKS ON GENERALIZED BOREL MEASURES IN TOPOLOGICAL SPACES

by

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Given a set $A$, we shall denote by $|A|$ the cardinality of $A$ and by $\exp A$ the family of all subsets of $A$. Throughout, $\alpha$ will denote an uncountable cardinal and $\beta$, $\gamma$, will denote infinite cardinals. Whenever convenient, we shall identify a cardinal with its initial ordinal. As usual, we shall denote by $\omega$ and $\Omega$ the first infinite and the first uncountable cardinals, respectively.

**Definition 1.** Let $M$ be a set. A family $\mathcal{M} \subset \exp M$ is called an $\alpha$-algebra in $M$ if

(i) $(A \in \mathcal{M}, \ |A| < \alpha) \Rightarrow \cup A \in \mathcal{M}$;

(ii) $A \in \mathcal{M} \Rightarrow M - A \in \mathcal{M}$.

It follows from (i) that $\emptyset \in \mathcal{M}$; thus by (ii), also $M \in \mathcal{M}$.

**Definition 2.** Let $\mathcal{M}$ be an $\alpha$-algebra in a set $M$. A function $\mu: \mathcal{M} \rightarrow [0, +\infty]$ is called an $\alpha$-measure on $\mathcal{M}$ if

$\mu(\emptyset) = 0$ and

$\mu(\cup A) = \sum \{\mu(A): A \in A\}$

for each disjoint family $A \subset \mathcal{M}$ with $|A| < \alpha$.

The triple $(M, \mathcal{M}, \mu)$ is called an $\alpha$-measure space.

**Definition 3.** Let $(M, \mathcal{M}, \mu)$ be an $\alpha$-measure space. The $\alpha$-measure $\mu$ is called $\beta$-finite if there is an $A \subset \mathcal{M}$ such that $|A| \leq \beta$, $\cup A = M$, and $\mu(A) < +\infty$ for each $A \in A$. 


If \( \alpha = \Omega \) and \( \beta = \omega \), then the previous definitions reduce to the usual definitions of a \( \sigma \)-algebra, measure, and a \( \sigma \)-finite measure. To illustrate the situation when \( \alpha > \Omega \) and \( \beta > \omega \), we shall present a few examples.

**Example 1.** Let \( M \) be a set and let \( f: M \to [0, +\infty] \). For \( A \subset M \) set

\[
\mu(A) = \sum \{ f(x) : x \in A \}.
\]

Then \( \mu \) is an \( \alpha \)-measure on \( \exp M \) for each \( \alpha \). If \( \mu(M) = +\infty \), then \( \mu \) is \( \beta \)-finite if and only if

\[
|\{ x \in M : f(x) > 0 \}| \leq \beta.
\]

**Example 2.** Let \( \kappa \geq \Omega \) be a regular ordinal and let \( W \) be the set of all ordinals less than \( \kappa \) equipped with the order topology. Denote by \( \mathcal{H} \) the family of all closed cofinal subsets of \( W \) and let \( \mathcal{M} \) consist of all sets \( A \subset W \) for which either \( A \) or \( W - A \) contain a set \( F \in \mathcal{H} \). For \( A \in \mathcal{M} \), let

\[
\mu(A) = 1 \text{ if } A \text{ contains a set } F \in \mathcal{H}, \text{ and } \mu(A) = 0 \text{ otherwise.}
\]

Clearly, \( \mathcal{M} \) contains all open subsets of \( W \). To show that \( (W, \mathcal{M}, \mu) \) is a \( \kappa \)-measure space, it suffices to prove the following claim.

**Claim.** If \( \mathcal{H}_0 \subset \mathcal{H} \) and \( |\mathcal{H}_0| < \kappa \), then \( \cap \mathcal{H}_0 \in \mathcal{H} \).

**Proof.** Using the interlacing lemma (see [8], chpt. 4, prbl. E, (a), p. 131) in \( W \), it is easy to prove the claim if \( |\mathcal{H}_0| = 2 \). By induction, the claim is correct if \( |\mathcal{H}_0| < \omega \).

Let \( \omega \leq \xi < \kappa \) and suppose that the claim holds whenever \( |\mathcal{H}_0| < \xi \). Let \( \mathcal{H}_0 = \{ F_\rho : \rho < \xi \} \). Replacing \( F_\rho \) by \( \cap \{ F_\tau : \tau \leq \rho \} \), we may assume that \( F_\tau \subset F_\rho \) for each \( \tau \leq \rho < \xi \). Choose an \( \eta < \kappa \). Since \( \kappa \) is regular, there are \( \eta_\tau \in F_\tau \) such that \( \eta < \eta_\tau < \eta_\rho \) for each \( \tau < \rho < \xi \). If
\[ \zeta = \sup \{ \eta_\rho : \rho < \xi \}, \]
then \( \zeta \in \bigcap H^0 \); for by the regularity of \( \kappa \), \( \zeta < \kappa \). It follows that \( \bigcap H^0 \) is a cofinal subset of \( W \) and the claim is proved.

**Example 3.** Let \( |M| = \kappa \) where \( \kappa \) is the first measurable cardinal (see Definition 6), and let \( \mu \) be a \( \sigma \)-additive measure on \( \exp M \) such that \( \mu(M) = 1 \) and \( \mu({x}) = 0 \) for each \( x \in M \). Then it follows from [2], Lemma 0.4.12 that \( \mu \) is a \( \kappa \)-measure on \( \exp M \).

**Example 4.** Karel Hrbacek kindly pointed out to me the following fact proved in [10], sec. 4, coroll. 1. If \( \alpha \) is a regular cardinal, then there is a model for the Zermelo-Fraenkel set theory with the axiom of choice in which

(i) \( 2^\omega = \alpha \) and Martin's axiom A holds;

(ii) The family \( L \) of all Lebesgue measurable subsets of reals is an \( \alpha \)-algebra and the Lebesgue measure is an \( \alpha \)-measure on \( L \).

Throughout, \( X \) will be a T\(_1\) space and \( \mathcal{G} \) will denote the family of all open subsets of \( X \). The intersection of all \( \alpha \)-algebras in \( X \) containing \( \mathcal{G} \) is again an \( \alpha \)-algebra in \( X \) containing \( \mathcal{G} \); it is denoted by \( \mathcal{B}_\alpha \) and called the Borel \( \alpha \)-algebra in \( X \). Clearly, \( \mathcal{B}_\omega \) is then the usual Borel \( \sigma \)-algebra in \( X \). An \( \alpha \)-measure \( \mu \) on \( \mathcal{B}_\alpha \) is called a Borel \( \alpha \)-measure in \( X \) if it is locally finite, e.g., if each \( x \in X \) has a neighborhood \( U \in \mathcal{B}_\alpha \) with \( \mu(U) < +\infty \).

A set \( A \subseteq X \) is called \( \gamma \)-Lindelöf if each open cover of \( A \) contains a subcover of cardinality less than \( \gamma \). Thus \( \Omega \)-Lindelöf sets are Lindelöf, and \( \omega \)-Lindelöf sets are compact.
The family of all \textit{closed} \( \gamma \)-Lindelöf subsets of \( X \) is denoted by \( \mathcal{J}_\gamma \). Clearly, \( \mathcal{J}_\gamma \) is the family of all closed subsets of \( X \) whenever \( \gamma > |X| \).

\textbf{Definition 4.} A Borel \( \alpha \)-measure in \( X \) is called

(i) \textit{diffused} if \( \mu(\{x\}) = 0 \) for each \( x \in X \);

(ii) \( \beta \)-\textit{moderated} if there is an \( A \subseteq \mathcal{G} \) such that \( |A| \leq \beta \),
\[ \bigcup A = X, \text{ and } \mu(A) < +\infty \text{ for each } A \in A; \]

(iii) \( \gamma \)-\textit{Radon} if
\[ \mu(A) = \sup\{\mu(F) : F \in \mathcal{J}_\gamma, F \subseteq A, \mu(F) < +\infty\} \]
for each \( A \in \mathcal{B}_\alpha \);

(iv) \( \gamma \)-\textit{regular} if it is \( \gamma \)-Radon and
\[ \mu(A) = \inf\{\mu(G) : G \in \mathcal{G}, A \subseteq G\} \]
for each \( A \in \mathcal{B}_\alpha \).

The Borel \( \alpha \)-measures which are \( \omega \)-\textit{moderated}, or \( \omega \)-\textit{Radon}, or \( \omega \)-\textit{regular} are usually called \textit{moderated}, or \textit{Radon}, or \textit{regular}, respectively.

\textbf{Discussion of results.} It is clear that a \( \beta \)-\textit{moderated} \( \alpha \)-measure is \( \beta \)-finite, and that a \( \beta \)-finite, \( \gamma \)-\textit{regular} \( \alpha \)-measure is \( \beta \)-\textit{moderated}. On the other hand, a \( \sigma \)-finite Radon measure is generally not moderated (see [4], ex. 7). It is easy to see that a moderated \( \gamma \)-Radon \( \alpha \)-measure is \( \gamma \)-\textit{regular}, yet the converse is false (e.g., giving \( M \) the discrete topology, the measure \( \mu \) from Example 1 is always regular, but not necessarily moderated). We shall show that for a large family of spaces each diffused \( \gamma \)-\textit{regular} \( \alpha \)-measure with \( \alpha \geq \gamma \) is moderated and hence \( \sigma \)-finite. Thus typically a non-\( \sigma \)-finite, \( \gamma \)-Radon \( \alpha \)-measure is not \( \gamma \)-\textit{regular}. We shall
prove, however, that a $\beta$-finite, $\gamma$-Radon $\alpha$-measure with $\alpha > \beta$ and $\alpha \geq \gamma$ is $\beta$-moderated whenever the space $X$ is meta-$\beta$-Lindelöf (see Definition 5, (i)). Under a mild set theoretic restriction, we shall also prove that in a metacompact space each $\beta$-finite, Borel $\alpha$-measure with $\alpha > \beta$ is $\beta$-moderated.

Sometimes many properties of a Borel $\alpha$-measure can be deduced from the well-known facts about the usual $\sigma$-additive Borel measures. To this end, we shall show that each moderated, $\gamma$-Radon $\alpha$-measure is a restriction of a complete Borel measure. We shall also prove that each $\gamma$-Radon $\alpha$-measure with $\alpha > \gamma^\omega$ degenerates to a measure from Example 1.

**Proposition 1.** Let $\alpha \geq \gamma$ and let $\mu$ be a $\gamma$-Radon $\alpha$-measure in $X$. Then there is a disjoint family $\mathcal{C} \subset \mathcal{F}_{\gamma}$ such that

(i) If $C \in \mathcal{C}$, then $C \neq \emptyset$, $\mu(C) < +\infty$, and $\mu(C \cap G) > 0$ for each $G \in \mathcal{G}$ with $C \cap G \neq \emptyset$;

(ii) If $B \in \mathcal{B}_\alpha$, then

$$\mu(B) = \sum \{\mu(B \cap C) : C \in \mathcal{C}\}.$$

**Proof.** By Zorn's lemma there is a maximal disjoint family $\mathcal{C} \subset \mathcal{F}_{\gamma}$ satisfying condition (i). Let $B \in \mathcal{B}_\alpha$. Since $\mathcal{C}$ is disjoint,

$$\mu(B) \geq \sum \{\mu(B \cap C) : C \in \mathcal{C}\}.$$

We shall prove the reverse inequality in three steps.

(a) Let $B \cap C = \emptyset$ for each $C \in \mathcal{C}$ and suppose that $\mu(B) > 0$. Then there is an $F \in \mathcal{F}_{\gamma}$ with $F \subset B$ and $0 < \mu(F) < +\infty$. If

$$H = \cup \{F \cap G : G \in \mathcal{G}, \mu(F \cap G) = 0\},$$

then $H$ is open in $F$ and $\mu(E) = 0$ for each $\gamma$-Lindelöf set.
E ∈ H. Since μ is γ-Radon, μ(H) = 0. Letting C₀ = F - H, we have C₀ ∈ \( J_γ \) and μ(C₀) = μ(F). In particular, C₀ ≠ ∅ and μ(C₀) < +∞. By the definition of H, μ(C₀ ∩ G) > 0 for each G ∈ \( J_γ \) with C₀ ∩ G ≠ ∅. It follows that \( C \cup \{ C₀ \} \) is a disjoint subfamily of \( J_γ \) satisfying condition (i); a contradiction.

(b) Let B ∈ \( J_γ \). By the local finiteness of μ, there is an open cover \( \mathcal{U} \) of B such that |\( \mathcal{U} \)| < γ and μ(U) < +∞ for each U ∈ \( \mathcal{U} \). Since C is disjoint, and since μ(C ∩ U) > 0 whenever C ∈ C, U ∈ \( \mathcal{U} \), and C ∩ U ≠ ∅, the family \( \{ C ∈ C : C ∩ U ≠ ∅ \} \) is countable for each U ∈ \( \mathcal{U} \). Thus

\[
|\{ C ∈ C : B ∩ C ≠ ∅ \}| < γ⋅ω = γ ≤ α,
\]

and by (a),

\[
μ(B) = Σμ(B ∩ C) : C ∈ C.
\]

(c) Let B ∈ \( β_α \). By (b), for each F ∈ \( J_γ \) with F ⊆ B,

\[
μ(F) = Σμ(F ∩ C) : C ∈ C ≤ Σμ(B ∩ C) : C ∈ C.
\]

Since μ is γ-Radon, also

\[
μ(B) ≤ Σμ(B ∩ C) : C ∈ C.
\]

In case of α = ω and γ = ω, a version of Proposition 1 was proved by N. Bourbaki and R. Godement, who called the family C a \( μ\)-concassage (see [13], p. 46).

A set A ⊆ X is called locally countable if each x ∈ X has a neighborhood U with |A ∩ U| ≤ ω.

Theorem 1. Let α > γ and let each uncountable locally countable set A ⊆ X contain an uncountable subset B ∈ \( β_α \). If μ is a diffused, γ-regular α-measure in X, then μ is moderated.

Proof. Let C be the family from Proposition 1. If C
is countable, then $\mu$ is $\sigma$-finite and thus by $\gamma$-regularity, also moderated. Hence assume that $C$ is uncountable, for each $C \in C$ choose $x_C \in C$, and let $A = \{x_C: C \in C\}$. It follows from Proposition 1 and the local finiteness of $\mu$ that $A$ is locally countable. According to our assumptions, we can find an uncountable set $B \subseteq A$ with $B \not= \emptyset$. Let $F \subseteq J_\gamma$ and $F \subseteq B$. Since $F$ can be covered by less than $\gamma$ open sets of finite measures, it follows from Proposition 1 that $|\{C \in C: C \cap F \not= \emptyset\}| < \gamma \cdot \omega = \gamma$. Consequently, $|F| < \gamma \leq \alpha$. Since $\mu$ is a diffused $\alpha$-measure, $\mu(F) = 0$. By the $\gamma$-regularity of $\mu$, $\mu(B) = 0$ and there is a $G \in \mathcal{G}$ such that $B \subseteq G$ and $\mu(G) < +\infty$. For each $x_C \in B$, we have $\mu(C \cap G) > 0$. Since $B$ is uncountable and $C$ is disjoint, this implies that $\mu(G) = +\infty$; a contradiction.

Next we shall show that the condition from Theorem 1 is satisfied by a large collection of familiar spaces.

Let $M$ be a set, and let $A \subseteq \text{exp} M$. For $x \in M$, set $o(x, A) = |\{A \in A: x \in A\}|$ and let $o(A)$ be the least cardinal such that $o(x, A) < o(A)$ for each $x \in M$. The cardinal $o(A)$ is called the order of $A$.

**Definition 5.** A space $X$ is called

(i) meta-$\beta$-Lindelöf if each open cover of $X$ has an open refinement $\mathcal{V}$ with $o(\mathcal{V}) \leq \beta$;

(ii) $\alpha$-weakly $\theta$-refinable if each open cover of $X$ has an open refinement $\mathcal{V} = \bigcup \{V_t: t \in T\}$ such that $|T| < \alpha$ and for each $x \in X$ there is a $t_x \in T$ with $1 \leq o(x, V_{t_x}) < \omega$. 

Clearly, meta-$\omega$-Lindelöf and meta-$\Omega$-Lindelöf spaces are, respectively, metacompact and meta-Lindelöf. Similarly, $\Omega$-weakly $\theta$-refinable spaces are weakly $\theta$-refinable in the sense of [1].

**Proposition 2.** Let $X$ be $\alpha$-weakly $\theta$-refinable and let $A \subset X$. If there is a $\beta < \alpha$ such that each $x \in X$ has a neighborhood $U$ with $|A \cap U| \leq \beta$, then $A \in \beta_\alpha$. In particular, if $A$ is locally countable, then $A \in \beta_\alpha$.

**Proof.** Suppose that there is a $\beta < \alpha$ such that each $x \in X$ has an open neighborhood $U_x$ with $|A \cap U_x| \leq \beta$. Let $V = \bigcup \{V_t : t \in T\}$ be an open refinement of $\{U_x : x \in X\}$ such that $|T| < \alpha$ and for each $x \in X$ there is a $t_x \in T$ with

$$l \leq o(x, V_{t_x}) < \omega.$$  

Since the sets $\{x \in X : o(x, V_t) \geq n\}$, $t \in T$, $n = 1, 2, \ldots$, are open, the sets

$$X_{t,n} = \{x \in X : o(x, V_t) = n\}$$

are Borel. Clearly,

$$X = \bigcup \{X_{t,n} : t \in T, n = 1, 2, \ldots\}.$$  

Let $W_{t,n}$ consist of all sets $X_{t,n} \cap V_1 \cap \cdots \cap V_n$ where $V_1, \ldots, V_n$ are distinct elements of $V_t$. Then $W_{t,n}$ is a disjoint family of open (in $X_{t,n}$) subsets of $X_{t,n}$ and $X_{t,n} = \bigcup W_{t,n}$. Moreover,

$$A \cap W = \{x_W^\rho : \rho < \kappa_W\}$$

where $\kappa_W \leq \beta$ for each $W \in W_{t,n}$. Thus the sets

$$A_{t,n,\rho} = \{x_W^\rho : W \in W_{t,n}, \kappa_W > \rho\},$$

$t \in T$, $n = 1, 2, \ldots$, $\rho < \beta$, are closed in $X_{t,n}$, and therefore Borel. We have
A = \bigcup\{A \cap X_{t,n}: t \in T, \ n = 1,2,\ldots\}
= \bigcup_{t \in T} \bigcup_{n=1}^{\infty} \bigcup\{A \cap W: W \in \mathcal{W}_{t,n}\}
= \bigcup_{t \in T} \bigcup_{n=1}^{\infty} \bigcup_{\rho < \beta} A_{t,n,\rho}.

Since |T| < \alpha, \omega < \alpha, \text{ and } \beta < \alpha, \text{ it follows that } A \in \beta_{\alpha}.

The following corollary is a direct consequence of Theorem 1 and Proposition 2.

**Corollary 1.** Let \( \alpha \geq \gamma \) and let \( X \) be \( \alpha \)-weakly \( \theta \)-refinable. If \( \mu \) is a diffused, \( \gamma \)-regular \( \alpha \)-measure in \( X \), then \( \mu \) is moderated.

Next two examples show that the assumptions of Proposition 2 are essential.

**Example 5.** For a regular ordinal \( \kappa \geq \Omega \), let \((W, \mathcal{M}, \mu_{\kappa})\) be the \( \kappa \)-measure space \((W, \mathcal{M}, \mu)\) from Example 2.

**Claim.** Let \( \kappa \geq \Omega \) be a regular ordinal. Then there is a set \( A_{\kappa} \subseteq W_{\kappa} \) such that \( A_{\kappa} \cap W_{\rho} \in \mathcal{M}_{\rho} \) for no regular ordinal \( \rho \in [\Omega, \kappa] \).

**Proof.** Since \( \Omega \) is not a measurable cardinal (see [14, thm. (A)]), there is a set \( A_{\Omega} \subseteq W_{\Omega} \) for which \( A_{\Omega} \notin \mathcal{M}_{\Omega} \). If \( \rho \in [\Omega, \kappa] \) is a regular ordinal, then each closed cofinal subset of \( W_{\rho} \) contains ordinals cofinal with both \( \omega \) and \( \Omega \). Thus it suffices to let \( A_{\kappa} \) be the union of \( A_{\Omega} \) and the set of all ordinals \( \zeta \in W_{\kappa} \) cofinal with \( \Omega \).

If the cardinal \( \kappa \) is an immediate successor of a cardinal \( \beta \), then \( |A_{\kappa} \cap [0,\zeta]| \leq \beta \) for each \( \zeta \in W_{\kappa} \) and yet \( A_{\kappa} \notin \mathcal{M}_{\kappa} \); for \( \beta_{\kappa} \subseteq \mathcal{M}_{\kappa} \). However, since each \( A \subseteq W_{\kappa} \) contains a discrete subset \( B \) with \( |B| = |A| \), Theorem 1 can be still applied to \( W_{\kappa} \).
Example 6. Let \( \kappa \) be a weakly inaccessible ordinal, i.e., \( \kappa \) is a regular ordinal and \( \kappa = \omega_\tau \) for some limit ordinal \( \tau \) (see [9], chpt. IX, sec. 1, p. 309). With the notation from Example 5, let

\[
X = \{ (\zeta, \eta) \in W_\kappa \times W_\kappa^*: \zeta \leq \eta \}
\]

where \( W_\kappa^* \) is the set \( W_\kappa \) with the discrete topology. Clearly, \( X \) is paracompact. If

\[
A = \{ (\zeta, \eta) \in X: \zeta \in A_\kappa \},
\]

then for each ordinal \( \rho \in [\Omega, \kappa) \),

\[
A \cap (W_{\rho+1} \times \{\rho\}) = (A_\kappa \cap W_{\rho+1}) \times \{\rho\}
\]

and consequently

\[
|A \cap (W_{\rho+1} \times \{\rho\})| < \kappa.
\]

By the claim in Example 5, \( A \in \beta_\alpha \) for no regular cardinal \( \alpha < \kappa \). It follows easily from the weak inaccessibility of \( \kappa \) that \( \cup_{\alpha < \kappa} \beta_\alpha \) is a \( \kappa \)-algebra in \( X \) containing all open sub-sets of \( X \). Therefore, \( \cup_{\alpha < \kappa} \beta_\alpha = \beta_\kappa \) and \( A \notin \beta_\kappa \).

As indicated by Example 5, the condition from Corollary 1 is not necessary. In fact, the following question seems open at this time.

**Question.** For \( \alpha \geq \gamma \), does there exist a diffused, \( \gamma \)-regular \( \alpha \)-measure which is not moderated?

**Lemma 1.** Let \( (M, \mathcal{M}, \mu) \) be an \( \alpha \)-measure space with \( \mu(M) < +\infty \), and let \( A \subset \mathcal{M} \). If \( \circ(A) < \alpha \) and \( \epsilon > 0 \), then

\[
|\{ A \in A: \mu(A) \geq \epsilon \}| < \max(\circ(A), \Omega).
\]

**Proof.** (a) Let \( \beta = \circ(A) \), \( \beta < \alpha \), and suppose that there is an \( \epsilon > 0 \) such that the set

\[
A_+ = \{ A \in A: \mu(A) \geq \epsilon \},
\]
has cardinality larger than or equal to \( \max(\beta, \Omega) \). First we shall show that there is a family \( \mathcal{C} \subseteq A_+ \) such that for each countable collection \( \mathcal{D} \subseteq \mathcal{C} \) we can find a \( C \in \mathcal{C} \) with \( \mu(C - \bigcup \mathcal{D}) > 0 \).

(b) If no such family exists, then for each \( \mathcal{C} \subseteq A_+ \) there is a countable \( \mathcal{C}_0 \subseteq \mathcal{C} \) such that \( \mu(C - \bigcup \mathcal{C}_0) = 0 \) for each \( C \in \mathcal{C} \). Let \( \mathcal{C}_1 = A_+ \) and define inductively \( \mathcal{C}_\tau, \tau < \beta \), by setting

\[
\mathcal{C}_\tau = A_+ - \bigcup_{\rho < \tau} \mathcal{C}_{\rho_{a_{\tau}}}
\]

where \( \mathcal{C}_{\rho_{a_{\tau}}} = (\mathcal{C}_{\rho})_{\beta} \). Since \( |\bigcup_{\rho < \tau} \mathcal{C}_{\rho_{a_{\tau}}}| \leq \tau \cdot \omega < \max(\beta, \Omega) \), the families \( \mathcal{C}_\tau \), and consequently \( \mathcal{C}_{\rho_{a_{\tau}}} \), are nonempty for each \( \tau < \beta \). If \( \mathcal{C}_\tau = \bigcup \mathcal{C}_{\rho_{a_{\tau}}} \), then \( \mu(C_{\tau}) \geq \varepsilon \) and \( \mu(C - C_{\tau}) = 0 \) for each \( C \in \mathcal{C}_\rho \) with \( \tau \leq \rho < \beta \). Thus \( \mu(C_{\rho} - C_{\tau}) = 0 \) whenever \( \tau < \rho < \beta \), and we obtain

\[
\mu(\bigcap_{\tau \leq \rho} C_{\tau}) = \mu(C_{\rho}) - \mu(C_{\rho} - \bigcap_{\tau \leq \rho} C_{\tau}) = \mu(C_{\rho}) - \mu(\bigcup_{\tau \leq \rho} (C_{\rho} - C_{\tau})) = \mu(C_{\rho}) > \varepsilon
\]

for each \( \rho < \beta \). Since \( \beta < \alpha \),

\[
\mu(\bigcap_{\tau < \beta} C_{\tau}) = \inf_{\rho < \beta} \mu(\bigcap_{\tau \leq \rho} C_{\tau}) > \varepsilon.
\]

In particular, \( \bigcap_{\tau < \beta} C_{\tau} \neq \emptyset \). If \( \mathcal{C}_{\rho_{a_{\tau}}} = \{A_{\tau}^k : k = 1, 2, \ldots\} \), then

\[
\bigcap_{\tau < \beta} C_{\tau} = \bigcap_{\tau < \beta} \bigcup_{k} A_{\tau}^k = \bigcup_{\{k(\tau)\}} \bigcap_{\tau < \beta} A_{\tau}^{k(\tau)}
\]

where the last union is taken over all transfinite sequences \( \{k(\tau)\}_{\tau < \beta} \) of positive integers. Thus there are positive integers \( k(\tau), \tau < \beta \), and an \( x \in M \) with \( x \in \bigcap_{\tau < \beta} A_{\tau}^{k(\tau)} \). Because the families \( \mathcal{C}_{\rho_{a_{\tau}}} \) are mutually disjoint, \( A_{\rho}^{k(\rho)} \neq A_{\tau}^{k(\tau)} \) whenever \( \rho \neq \tau \). Consequently \( o(x, \mathcal{A}) = \beta \), and this contradiction establishes the existence of the family \( \mathcal{C} \) from (a).

(c) Choose \( E_0 \in \mathcal{C} \) and suppose that for each \( \rho < \tau < \Omega \) we have chosen \( E_{\rho} \in \mathcal{C} \) so that
\[ \mu(E_\rho - \bigcup_{\lambda < \rho} E_\lambda) > 0. \]

Since \( \{E_\rho: \rho < \tau\} \) is a countable subfamily of \( C \), there is an \( E_\tau \in C \) such that
\[ \mu(E_\tau - \bigcup_{\rho < \tau} E_\rho) > 0. \]

Letting \( F_\tau = E_\tau - \bigcup_{\rho < \tau} E_\rho \), we obtain an uncountable disjoint family \( \{F_\tau: \tau < \Omega\} \subset M \) of sets with positive measures. It follows that \( \mu(M) = +\infty \); a contradiction.

**Corollary 2.** Let \((M, \mathcal{M}, \mu)\) be an \( \alpha \)-measure space with \( \mu(M) < +\infty \), and let \( A \subset \mathcal{M} \). If \( o(A) < \alpha \) and \( \epsilon > 0 \), then
\[ |\{A \in A: \mu(A) \geq \epsilon\}| < \max(o(A), \omega). \]

**Proof.** In view of Lemma 1, it suffices to consider the case of \( o(A) \leq \omega \). This implies that the set
\[ A_+ = \{A \in A: \mu(A) \geq \epsilon\} \]
is countable. If \( \chi_A \) is the characteristic function of a set \( A \subset M \), then
\[ \Sigma\{\chi_A(x): A \in A_+\} = o(x, A_+) \]
for each \( x \in M \). Since \( A_+ \) is countable, the function \( x \mapsto o(x, A_+) \) is measurable, and so are the sets
\[ M_n = \{x \in M: o(x, A_+) \leq n\}, \]
\( n = 1, 2, \cdots \). The sequence \( \{M_n\} \) is increasing and \( \bigcup_{n=1}^{\infty} M_n = M \); for if \( x \in M \), then
\[ o(x, A_+) < o(A) \leq \omega. \]
Thus there is an integer \( p \geq 1 \) such that \( \mu(M - M_p) < \frac{\epsilon}{2} \). We have
\[ \Sigma\{\mu(A \cap M_p): A \in A_+\} = \Sigma\{\int_{M_p} \chi_A \ d\mu: A \in A_+\} \leq \int_{M_p} p \ d\mu \]
\[ = p \mu(M_p) < +\infty. \]
Because \( \mu(A \cap M_p) \geq \frac{\epsilon}{2} \) for each \( A \in A_+ \), it follows that \( |A_+| < \omega. \)
Example 7. Let $\kappa > \Omega$ be a regular ordinal and let $(W, M, u)$ be the $\kappa$-measure space from Example 2. If $A = \{(\rho, \kappa): \rho < \kappa\}$, then $\mu(A) = 1$ for each $A \in A$ and $o(A) = |A| = \kappa$.

Proposition 3. Let $\alpha > \beta$ and let $(M, M_1, u)$ be an $\alpha$-measure space with $\mu \beta$-finite. If $A \subseteq M$ and $o(A) \leq \beta$, then $|\{A \in A: \mu(A) > 0\}| \leq \beta$.

Proof. Let $M = \cup C$ where $C \subseteq M$, $|C| \leq \beta$, and $u(C) < +\infty$ for each $C \in C$. For $C \in C$ and $A \in M$, let $\mu_C(A) = \mu(A \cap C)$.

Clearly, $\mu_C$ is a finite $\alpha$-measure on $M$ and

$$\mu(A) = \sum_{C \in C} \mu_C(A): C \in C$$

for every $A \in M$. If

$$A_C = \{A \in A: \mu_C(A) > 0\},$$

then

$$A_C = \cup_{n=1}^{\infty} \{A \in A: \mu_C(A) > \frac{1}{n}\}$$

and so by Corollary 2, $|A_C| \leq \omega \cdot \beta = \beta$. Since

$$\{A \in A: \mu(A) > 0\} = \cup \{A_C: C \in C\},$$

the proposition follows.

Letting $\alpha = \Omega$ in Proposition 3, we obtain the following corollary.

Corollary 3. Let $(M, M, u)$ be a measure space with a $\sigma$-finite measure $\mu$. If $A \subseteq M$ is a point-finite family, then $\mu(A) = 0$ for all but countably many $A \in A$.

Note. Without proof, Corollary 3 was first communicated to me by Heikki Junnila. Although quite analogous, his proof of Corollary 3 (see [7]) and my proof of Lemma 1 were obtained independently. There is a simple direct proof of Corollary 3
(see [12], chpt. 18, ex. (18-18)), which was found jointly by Don Chakerian and myself.

**Theorem 2.** Let $\alpha > \beta$, $\alpha \geq \gamma$, and let $\mu$ be a $\beta$-finite, $\gamma$-Radon $\alpha$-measure in $X$. If $X$ is meta-$\beta$-Lindelöf, then $\mu$ is $\beta$-moderated.

**Proof.** If $X$ is meta-$\beta$-Lindelöf, then there is an open cover $\mathcal{A}$ of $X$ such that $\sigma(\mathcal{A}) \leq \beta$ and $\mu(U) < +\infty$ for each $U \in \mathcal{A}$. By Proposition 3,

$$|\{U \in \mathcal{A} : \mu(U) > 0\}| \leq \beta.$$  

Thus if $\mathcal{A}_0 = \{U \in \mathcal{A} : \mu(U) = 0\}$ and $\mathcal{G} = \cup \mathcal{A}_0$, it suffices to show that $\mu(\mathcal{G}) < +\infty$. Let $F \in J_\gamma$ and $F \subset \mathcal{G}$. There is $V \subset \mathcal{A}_0$ such that $|V| < \gamma$ and $F \subset \cup V$. It follows that $\mu(F) = 0$ and since $\mu$ is $\gamma$-Radon, also $\mu(\mathcal{G}) = 0$.

**Definition 6.** A cardinal $\kappa$ is called measurable if there is a discrete space $Y$ of cardinality $\kappa$ and a diffused, Borel $\kappa$-measure $\mu$ in $Y$ with $\mu(Y) = 1$.

The basic properties of measurable cardinals which do not involve axiomatic set theory are proved in [14]; more recent results can be found, e.g., in [2], chpt. 0, sec. 4.

The next lemma is proved by a modified technique of Haydon (see [6], Prop. 3.2).

**Lemma 2.** Let $\alpha > \beta$ and let $\mu$ be a $\beta$-finite, Borel $\alpha$-measure in $X$. Let $\mathcal{A} \subset \mathcal{G}$ be a point-finite family such that $\mu(U) = 0$ for each $U \in \mathcal{A}$. If $X$ contains no discrete subspace of measurable cardinality, then $\mu(\cup \mathcal{A}) = 0$.

**Proof.** Let $X = \cup \mathcal{C}$ where $\mathcal{C} \subset \beta_\alpha$, $|\mathcal{C}| \leq \beta$, and $\mu(C) < +\infty$ for each $C \in \mathcal{C}$. Because the sets $\{x \in X : \sigma(x, \mathcal{A}) \geq n\}$,
n = 1, 2, ..., are open, the sets

\[ X_n = \{ x \in X : \sigma(x, A) = n \} \]

are Borel. Clearly \( \bigcup X_n = \bigcup_{n=1}^{\infty} X_n \), and so it suffices to show that \( \mu(C \cap X_n) = 0 \) for each \( C \in \mathcal{C} \) and \( n = 1, 2, \ldots \). Fix a \( C \in \mathcal{C} \) and an integer \( n \geq 1 \), and suppose that \( \mu(C \cap X_n) > 0 \).

Consider the family \( V \) of all nonempty sets

\[ V = C \cap X_n \cap U_1 \cap \cdots \cap U_n \]

where \( U_1, \ldots, U_n \) are distinct elements of \( A \). Since \( V \) is a disjoint open (in \( C \cap X_n \)) cover of \( C \cap X_n \), we can define an \( \alpha \)-measure \( \nu \) on \( \exp V \) by letting

\[ \nu(V') = \frac{\mu(U' \cap X_n)}{\mu(C \cap X_n)} \]

for each \( V' \subset V \). It follows from [2], lemma 0.4.12 that \( V \) contains a family \( V_o \) of measurable cardinality. Choosing \( x_v \in V \) for each \( V \in V_o \) we obtain a discrete subspace \( X_o = \{ x_v : V \in V_o \} \) with \( |X_o| = |V_o| \); a contradiction.

**Theorem 3.** Let \( \alpha > \beta \) and let \( \mu \) be a \( \beta \)-finite, Borel \( \alpha \)-measure in \( X \). If \( X \) is metacompact and contains no discrete subspace of measurable cardinality, then \( \mu \) is \( \beta \)-moderated.

**Proof.** Choose a point-finite open cover \( A \) of \( X \) such that \( \mu(U) < +\infty \) for each \( U \in A \). By Proposition 3,

\[ | \{ U \in A : \mu(U) > 0 \} | < \beta, \]

and by Lemma 2,

\[ \mu( \{ U \in A : \mu(U) = 0 \} ) = 0. \]

The theorem follows.

We do not know whether Theorem 3 remains correct if the assumption "\( X \) contains no discrete subspace of measurable cardinality" is relaxed to "\( X \) contains no closed discrete
subspace of measurable cardinality." However, using techniques of Moran (see [11], prop. 4.2) one can show easily that in Theorem 3, instead of assuming that $X$ contains no discrete subspace of measurable cardinality, we may assume that all closed discrete subspaces of $X$ have semi-reducible cardinality (for the definition and basic properties of semi-reducible cardinals see [11], sec. 3).

Note. Using a slightly different technique, Theorems 2 and 3 were proved in [4] for $\alpha = \Omega$ (see [4], Lemma 3 and Remark 2).

Let $\mu$ be a Borel $\alpha$-measure. Since the Borel $\sigma$-algebra $\beta = \beta_\Omega$ is contained in $\beta_\alpha$, we can define a measure space $(X, \overline{\beta}, \overline{\mu})$ as the usual completion of the measure space $(X, \beta, \mu)$ (see [5], sec. 13, p. 55). The measure space $(X, \overline{\beta}, \overline{\mu})$ is said to be associated with the Borel $\alpha$-measure $\mu$.

Proposition 4. Let $\mu$ be a Borel $\alpha$-measure in $X$ and let $(X, \overline{\beta}, \overline{\mu})$ be the measure space associated with $\mu$. If $\mu$ is moderated and $\gamma$-Radon with an arbitrary $\gamma$, then $\beta_\alpha \subset \beta$ and $\overline{\mu}(A) = \mu(A)$ for each $A \in \beta_\alpha$.

Proof. (a) Let $A \in \beta_\alpha$ and suppose that $A \subset H$ for some $H \in \mathcal{G}$ with $\mu(H) < +\infty$. Then
\[
\mu(A) = \sup\{\mu(F) : F \in \mathcal{J}_\gamma, F \subset A\} = \inf\{\mu(G) : G \in \mathcal{G}, A \subset G\}.
\]
Thus there is an $F_\delta$ set $F_\delta$ and a $G_\delta$ set $G_\delta$ such that $F_\delta \subset A \subset G_\delta$ and $\mu(F_\delta) = \mu(A) = \mu(G_\delta) < +\infty$. It follows that $A = F_\delta \cup (A - F_\delta)$ belongs to $\overline{\beta}$ and that $\overline{\mu}(A) = \overline{\mu}(F_\delta) = \mu(F_\delta) = \mu(A)$.
(b) Let $A \in \mathcal{B}_\alpha$ be arbitrary. Since $\mu$ is moderated, 
$X = \bigcup_{n=1}^{\infty} B_n$ where $H_n \in \mathcal{G}$ and $\mu(H_n) < +\infty$, $n = 1, 2, \ldots$. By (a), $A \cap H_n \in \mathcal{B}$ and $\overline{\mu}(A \cap H_n) = \mu(A \cap H_n)$, $n = 1, 2, \ldots$. Consequently, $A \in \mathcal{B}$ and $\overline{\mu}(A) = \mu(A)$.

The following theorem generalizes an unpublished result of Gary Gruenhage.

**Theorem 4.** Let $\mu$ be a $\gamma$-Radon $\alpha$-measure in $X$. If $\alpha > \prod_{n=1}^{\infty} \gamma_n$ whenever $\gamma_n < \gamma$, $n = 1, 2, \ldots$, then

$$\mu(B) = \sum_{x \in B} \mu(\{x\})$$

for each $B \in \mathcal{B}_\alpha$.

**Proof.** (a) Let $X$ be $\gamma$-Lindelöf and let $\mu(X) < +\infty$. For each $x \in X$ and $n = 1, 2, \ldots$, choose an $F_{x,n} \in \mathcal{G}$ such that

$$F_{x,n} \subseteq X - \{x\}$$

and

$$\mu(F_{x,n}) > \mu(X - \{x\}) - \frac{1}{n}.$$  

Each open cover $\{X - F_{x,n} : x \in X\}$ of $X$ has a subcover $U_n$ with 

$$|U_n| < \gamma,$$ 

$n = 1, 2, \ldots$. Since $\mu(X) < +\infty$, the set $A = \{x \in X : \mu(\{x\}) > 0\}$ is countable, and since

$$\mu(X - F_{x,n}) < \mu(\{x\}) + \frac{1}{n}$$

for each $x \in X$, $\mu(U - A) < \frac{1}{n}$ for each $U \in U_n$. Thus

$$\mu(\cap_{n=1}^{\infty} (U_n - A)) = 0$$

for every sequence $\{U_n\}$ with $U_n \in U_n$, $n = 1, 2, \ldots$. We have

$$X - A = \cap_{n=1}^{\infty} \cup\{U - A : U \in U_n\} = \cup\{U_n\} \cap_{n=1}^{\infty} (U_n - A)$$

where the last union is taken over all sequences $\{U_n\}$ with $U_n \in U_n$, $n = 1, 2, \ldots$. Because $|U_n| < \gamma$, the collection of all these sequences has the cardinality less than $\alpha$. Consequently, $\mu(X - A) = 0$ and
\[
\mu(B) = \mu(A \cap B) = \Sigma\{\mu(\{x\}) : x \in A \cap B\}
\]
\[
= \Sigma\{\mu(\{x\}) : x \in B\}
\]

for each \(B \in \beta^\alpha\).

(b) Let \(X\) and \(\mu\) be arbitrary, and let \(B \in \beta^\alpha\). There are \(F_n \in \mathcal{F}_\gamma\) such that \(F_n \subset F_{n+1} \subset B\), \(\mu(F_n) < +\infty\), \(n = 1, 2, \ldots\), and \(\lim \mu(F_n) = \mu(B)\). Using (a), we obtain
\[
\mu(B) = \lim \Sigma\{\mu(\{x\}) : x \in F_n\}
\]
\[
= \Sigma\{\mu(\{x\}) : x \in \bigcup_{n=1}^{\infty} F_n\} \leq \Sigma\{\mu(\{x\}) : x \in B\}
\]
The equality holds trivially when \(\mu(B) = +\infty\). If \(\mu(B) < +\infty\), then \(\mu(B - \bigcup_{n=1}^{\infty} F_n) = 0\) and the equality holds again.

Remark. The cardinality assumption in Theorem 4 is clearly satisfied when \(\alpha > \gamma^\omega\). However, this later condition is generally stronger. For example, choose infinite cardinals \(\kappa^\rho\) so that \(2^{\kappa^\rho} < 2^{\kappa^{\tau}}\) for each \(\rho < \tau < \omega\), and let \(\gamma = \sup\{2^{\kappa^\rho} : \rho < \omega\}\). If \(\gamma_n < \gamma\), \(n = 1, 2, \ldots\), then \(\gamma_n \leq 2^{\kappa^\rho}\) for some \(\rho < \omega\), and consequently
\[
\prod_{n=1}^{\infty} \gamma_n \leq 2^{\kappa^\rho \omega} = 2^{\omega \cdot \kappa^\rho} = 2^{\kappa^\rho} < \gamma \leq \gamma^\omega.
\]

We shall close this paper by stating two theorems about \(\alpha\)-measures which for \(\alpha = \omega\) were proved previously by Gardner, Gruenhage, and the author (see [3], corollary to Theorem 6.1, and [4], Theorem 2). Recall that a space \(X\) is:

(i) **hereditarily \(\alpha\)-weakly \(\theta\)-refinable** if each subspace of \(X\) is \(\alpha\)-weakly \(\theta\)-refinable;

(ii) **locally \(\gamma\)-Lindelöf** if each \(x \in X\) has a \(\gamma\)-Lindelöf neighborhood.

**Theorem 5.** Suppose that \(X\) is a regular, hereditarily \(\alpha\)-weakly \(\theta\)-refinable space which contains no discrete subspace
of measurable cardinality. Let $\alpha > \beta$ and let $\mu$ be a $\beta$-finite, Borel $\alpha$-measure in $X$. Then $\mu$ is $\gamma$-Radon for each $\gamma > \lvert X \rvert$.

Theorem 6. Suppose that $X$ is a regular, $\alpha$-weakly $\theta$-refinable, locally $\gamma$-Lindelöf space which contains no discrete subspace of measurable cardinality. Let $\alpha > \beta$ and let $\mu$ be a $\beta$-finite, Borel $\alpha$-measure in $X$. Then $\mu$ is $\gamma$-Radon if and only if it is $\delta$-Radon for some $\delta > \omega$.

Modulo the obvious adjustments, for a finite $\alpha$-measure $\mu$ the proofs of Theorems 5 and 6 are the same as those of Theorem (18.31) in [12] and Theorem 2 in [4], respectively. Since each $\beta$-finite $\alpha$-measure with $\alpha > \beta$ is a sum of finite $\alpha$-measures (see the proof of Proposition 3), it suffices to observe that a sum of $\gamma$-Radon $\alpha$-measures is also $\gamma$-Radon.

References

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