BOXES OF COMPACT ORDINALS

by

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If \( \{X_n : n \in \omega \} \) is a family of spaces, then \( \Box_{n \in \omega} X_n \), called the box product of those spaces, denotes the cartesian product of the sets with the topology generated by all sets of the form \( \prod_{n \in \omega} G_n \), where each \( G_n \) need only be open in the factor space \( X_n \). If \( X_n = X \forall n \in \omega \), we denote \( \Box_{n \in \omega} X_n \) by \( \Box \omega X \).

M. E. Rudin [5] and K. Kunen [3 and 6, pg. 58] have shown that CH implies \( \bigcup_{n \in \omega} (\lambda_n + 1) \) is paracompact for every countable collection of ordinals \( \{\lambda_n : n \in \omega \} \). At the 1976 Auburn University Topology Conference I demonstrated [7] that the paracompactness of \( \bigcup \omega (\omega + 1) \) is implied by the existence of a \( k \)-scale in \( \omega \), a set-theoretic axiom which is a consequence of, but not equivalent to, Martin's Axiom, and hence CH. In addition, I proved \( \bigcup \omega (\omega_1 + 1) \) is paracompact iff \( \bigcup \omega (\alpha + 1) \) is paracompact \( \forall \) countable ordinals \( \alpha \). If this is coupled with E. van Douwen's (\( \exists \) a \( k \)-scale in \( \omega \) \( \Rightarrow \) \( n \in \omega X_n \) is paracompact for all collections \( \{X_n : n \in \omega \} \) of compact metrizable spaces [1], we have \( \bigcup \omega (\omega_1 + 1) \) is paracompact if \( \exists \) a \( k \)-scale in \( \omega \omega \). However, none of the proofs generalize to higher ordinals (\( \bigcup \omega (\omega_2 + 1) \), for example). We conjecture:

If \( \bigcup \omega (\omega + 1) \) is paracompact, then \( \bigcup \omega (\lambda + 1) \) is paracompact \( \forall \) ordinals \( \lambda \).

\(^1\)It is unknown whether it is consistent for \( \bigcup \omega (\omega + 1) \) not to be paracompact; however, \( \exists \) compact spaces \( X_n \) such that \( \bigcup_{n \in \omega} X_n \) is not normal. Moreover, irrationals \( x(\bigcup \omega (\omega + 1)) \) is not normal [6, pg. 58].
Toward this conjecture we show:

Suppose $\lambda$ is an ordinal for which $\bigcup_{n \in \omega} (\lambda_n + 1)$ is para-
compact whenever $\lambda_n < \lambda \land n \in \omega$, then $\bigcup_{n \in \omega} (\lambda + 1)$ if either of
the following holds:

1. $\text{cf}(\lambda) \neq \omega$ (Theorem 1).
2. $\text{cf}(\lambda) = \omega$ and $\exists$ a $\kappa$-scale in $\omega$ (Theorem 2).

Now suppose $\{X_n : n \in \omega\}$ is a family of sets and for each $f \in \prod_{n \in \omega} X_n$,

$$E(f) = \{g \in \prod_{n \in \omega} X_n : (\exists m \in \omega) n > m \Rightarrow g(n) = f(n)\},$$

then $\{E(f) : f \in \prod_{n \in \omega} X_n\}$ forms a partition of $\prod_{n \in \omega} X_n$ and the
resultant quotient set is denoted by $\bigvee_{n \in \omega} X_n$. If $S \subseteq \prod_{n \in \omega} X_n$,
we let $E(S)$ denote its image in $\bigvee_{n \in \omega} X_n$.

Lemma (Kunen [3 and 6, pg. 58]). Suppose $X_n$ is a compact
Hausdorff space for each $n \in \omega$ and $\bigvee_{n \in \omega} X_n$ has the quotient
topology induced by $\bigcup_{n \in \omega} X_n$, then

1. $G_\delta$-sets in $\bigvee_{n \in \omega} X_n$ are open
2. $\bigcup_{n \in \omega} X_n$ is paracompact iff $\bigvee_{n \in \omega} X_n$ is paracompact
3. Every open cover of $\bigvee_{n \in \omega} X_n$ has a subcover of cardinality $\leq \mathfrak{c}$ (the cardinality of the continuum)
   whenever $X_n$ is scattered $\forall n \in \omega$.

For $A, B \in \mathcal{P}(\omega)$ define $A \leq B$ if $A - B$ is finite; $A \equiv B$
if $A \leq B$ and $B \leq A$. Observe that $\equiv$ is an equivalence relation
on $\mathcal{P}(\omega)$. Suppose $\lambda$ is an ordinal and $f \in \omega^\lambda$, for each
$A \in \mathcal{P}(\omega)$, we define in $\bigvee_{n \in \omega} (\lambda + 1)$, $\langle A, f \rangle = E(\prod_{n \in \omega} A f(n))$, where

2With (i) $\bigvee_{n \in \omega} X_n$ is paracompact iff every open cover has a
pairwise disjoint clopen refinement.
Theorem 1. Suppose \( \lambda \) is an ordinal with \( \text{cf}(\lambda) \neq \omega \), then for \( \mathcal{U}^\omega(\lambda + 1) \) to be paracompact it is necessary and sufficient that \( \mathcal{U}^\omega(\alpha + 1) \) be paracompact for every \( \alpha < \lambda \).

Proof. Necessity is obvious so we prove sufficiency only.

Without loss of generality, we assume \( \lambda \) is the supremum of an increasing sequence \( \{ n_\alpha : \alpha < \text{cf}(\lambda) \} \). Let \( R \) be an open cover of \( \mathcal{U}^\omega(\lambda + 1) \). For each \( \tau < \omega_1 \) and \( d \in \tau \) we construct inductively \( V(d) \), \( W(d) \), \( \theta(d) \), and \( A(d) \) to satisfy:

1. \( V(d) \) and \( W(d) \) are clopen subsets of \( \mathcal{U}^\omega(\lambda + 1) \), \( \exists U \in R \ni V(d) \subseteq U, V(d) \cup W(d) \subseteq W(d \uparrow \sigma) \) \( \forall \sigma < \tau \), and if \( \sigma < \tau \) is a limit ordinal, then \( W(d \uparrow \sigma) = \bigcap_{\rho < \sigma} W(d \uparrow \rho) \).
2. If \( \sigma \leq \tau \) is an odd ordinal \( ^3 \), then
   \[ \{ V(e) : \text{dom}(e) \leq \sigma \} \cup \{ W(e) : \text{dom}(e) = \sigma \} \]
   is a pairwise-disjoint covering of \( \mathcal{U}^\omega(\lambda + 1) \).
3. \( A(d) \) is an infinite subset of \( \omega \) and if \( \sigma \leq \tau \) is a non-limit ordinal, then \( A(d \uparrow \sigma) \leq A(d \uparrow \rho) \) \( \forall \rho < \sigma \).
4. If \( E(x) \in W(d) \) and \( \phi < A \leq A(d \uparrow \sigma) \) \( \forall \sigma \leq \tau \), then \( E(\{ y : x(n) \leq y(n) \leq \lambda \text{ if } n \in A, y(n) = x(n) \text{ if } n \notin A \}) \subseteq W(d) \).
5. \( \theta(d) \in \omega \lambda \) is a constant function with values in \( \{ n_\alpha : \alpha < \text{cf}(\lambda) \} \) and if \( \sigma \leq \tau \) is even, then
   \[ \theta(d \uparrow \sigma)(0) > \theta(d \uparrow \rho)(0) \] \( \forall \rho < \sigma \).

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\(^3\sigma \) is an odd ordinal when \( \sigma = \sigma_0 + 2n + 1 \), where \( \sigma_0 = 0 \) or is a limit ordinal and \( n \in \omega \). If \( \sigma \) is not odd it is even.
(6) If $\sigma \leq \tau$ is odd, then $W(d \uparrow \sigma) \leq \left< A(d \uparrow \sigma), \theta(d \uparrow \sigma) \right>$, 

(7) If $\sigma \leq \tau$ is a non-limit even ordinal and $\rho = \sigma - 1$, then

\exists a clopen subset $G(d \uparrow \sigma)$ of $\bigvee_{n \in A(d \uparrow \rho)} (\theta(d \uparrow \rho)(n) + 1)$ such that

$$V(d \uparrow \sigma) = W(d \uparrow \sigma) \cap \left< A(d \uparrow \rho), \theta(d \uparrow \sigma) \right>$$ and

$$W(d \uparrow \sigma) = \{ E(x) \in W(d \uparrow \rho) : E(x \uparrow \omega - A(d \uparrow \rho)) \in G(d \uparrow \sigma) \}.$$ 

Now suppose our objects $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ have been constructed to satisfy (1) through (7) $\forall d \in \tauC \forall \tau < \omega_1$. If $E(x) \notin \bigcup \{ V(t \uparrow \tau) : t \in \omega_1, \tau < \omega_1 \}$ then by (1) and (2) we may find for each $\tau < \omega_1$, $d_\tau \in \tauC$ such that $E(x) \in W(d_\tau)$. Again from (1) and (2), if $\sigma < \tau$ is odd and $d_\sigma \in \tauC$ such that $d \neq d_\sigma \uparrow \sigma$, then $E(x) \notin W(d)$; therefore, $\sigma < \tau \Rightarrow d_\sigma = d_\tau \uparrow \sigma$. From (5) we may find the first even ordinal $\rho < \omega_1$ such that for every $n$,

$$x(n) > \theta(d_\rho)(0) \Rightarrow x(n) \geq \sup_{\tau < \omega_1} \theta(d_\tau)(0).$$

From (6) $\exists y \in \bigcup \omega(\lambda + 1) \exists E(y) = E(x)$ and

$$A(d_{\rho + 1}) = \{ n : y(n) > \theta(d_{\rho + 1})(n) \}.$$ 

From (7) $E(y) \in V(d_{\rho + 2})$, a contradiction. Therefore,

$$\{ V(t \uparrow \tau) : t \in \omega_1, \tau < \omega_1 \}$$

is a cover of $\bigcup \omega(\lambda + 1)$ and we are done, so we should begin our construction.

Let $A(\phi) = \omega$, $W(\phi) = \bigcup \omega(\lambda + 1)$, and $\alpha$ be the first ordinal such that $E(\Pi^\omega[\alpha, \lambda])$ is contained in some $U \in R$. Let $\theta(\phi)(n) = n_\alpha \forall n \in \omega$ and $V(\phi) = \left< A(\phi), \theta(\phi) \right>$.

Suppose for an ordinal $\rho < \omega_1$ we have constructed $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ to satisfy (1) through (7) $\forall d \in \tauC \forall \tau < \rho$. Our construction at $\rho$ needs three cases:

Case 1. $\rho$ is an odd ordinal

Let $\tau = \rho - 1$ and $\theta(e) = \theta(d)$ if $e \in \tauC$ and $e \uparrow \tau = d$. Let
be a listing of exactly one element chosen from each equivalence class of elements of
\[ \{ A : \phi < A < A(\uparrow \sigma), \sigma \leq \tau \} \].

For each \( e \in \rho_c \) we let
\[ W(e) = W(e \uparrow \tau) \cap \{ A(e), \theta(e) \} \].

If \( d \in \tau_c \), then \( W(d) \cap \{ \phi, \theta(d) \} \) is a clopen subset of
\( E(\Pi \omega[0, \theta(d)(0)]) \); therefore, by the lemma (ii) and (iii) we may find a pairwise-disjoint clopen refinement of \( R \)
\[ \{ V(e) : e \in \rho_c, e \uparrow \tau = d \} \) whose union is \( W(d) \cap \{ \phi, \theta(d) \} \).

Clearly (1) through (7) are satisfied.

Case 2. \( \rho \) is a non-limit even ordinal.

Let \( \tau = \rho - 1 \) and \( A(e) = A(d) \) if \( e \in \rho_c \) and \( e \uparrow \tau = d \). If
\( d \in \tau_c \) and \( W(d) = \phi \), we let \( W(e) = V(e) = \phi \) and
\[ \theta(e)(n) = n \alpha \text{ if } \theta(d)(n) = n \alpha \quad \forall n \in \omega \]
If \( d \in \tau_c \) and \( W(d) \neq \emptyset \), let
\[ Y^*(d) = \{ g : g^{-1}(\lambda) = A(d), E(g) \in W(d) \} \].

We will wish to cover \( Y^*(d) \) by
\[ \cup \{ W(e) : e \uparrow \tau = d \} \).

From (4), \( Y(d) = \{ g \uparrow \omega - A(d) : g \in Y^*(d) \} \neq \emptyset \).

In \( \forall n \notin A(d) (\theta(d)(n) + 1) \), let
\[ R(d) = \{ E(\Pi n \notin A(d) U(n)) : E(\Pi n \in \omega U(n)) \subseteq \text{some } U \in R, \]
\[ E(\Pi U(n)) \cap Y^*(d) \neq \emptyset \} \).

From (5) of the induction hypothesis and the lemma, (ii) and (iii), \( \exists \) a pairwise disjoint clopen refinement \( \{ G(\gamma) : \gamma < \varsigma \} \)
of \( R(d) \) whose union is \( E(Y(d)) \). If \( e \in \rho_c \), \( e \uparrow \tau = d \), \( e(\tau) = \gamma \),
then let
\[ W(e) = \{ E(x) \in W(d) : E(x \uparrow \omega - A(d)) \in G(\gamma) \} \].
For each \( \gamma \) we may find \( n_\alpha(\gamma) > \emptyset(d)(0) \) such that
\[
\{E(x) \in W(d) : E(x \uparrow \omega - A(d)) \in G(\gamma) \text{ and } x(n) > n_\alpha(\gamma) \} \forall \text{ but finitely many } n \in A(d) \subseteq \text{some } U \in R.
\]
Let \( \emptyset(e)(n) = n_\alpha(\gamma) \) \( \forall n \in \omega \) and \( V(e) = W(e) \cap \langle A(d), \emptyset(e) \rangle \).
Certainly (1) through (7) are satisfied.

**Case 3.** \( \rho \) is a limit ordinal.

If \( e \in \mathcal{P} \), let \( A(e) = \omega \), \( V(e) = \emptyset \), and find the first \( \alpha < \omega_1 \ni n_\alpha > \emptyset(e \uparrow \tau)(0) \) \( \forall \tau < \rho \). We choose \( \emptyset(e)(n) = n_\alpha \) \( \forall n \in \omega \). To satisfy (1) through (7) we observe that (i) of the lemma allows
\[
W(e) = \bigcap_{\tau < \rho} W(e \uparrow \tau)
\]
to be clopen.

The proof to Theorem 1 is completed.

If \( \omega_1 \) is ordered by \( f < g \) if \( \{n : g(n) \leq f(n)\} \ni \emptyset \), then for an ordinal \( k \), a \( k \)-scale is an order-preserving injection \( s : k \rightarrow \omega_1 \) such that \( \{s(\alpha) : \alpha < k\} \) is cofinal in \( \omega_1 \). Recall [2,7] that \( \text{CH} \rightarrow \exists \) an \( \omega_1 \)-scale; \( \text{MA} \rightarrow \exists \) a \( c \)-scale; an \( \omega \)-scale; \( \exists \) a \( k \)-scale and \( \lambda \)-scale \( \Rightarrow \text{cf}(k) = \text{cf}(\lambda) \); for every model \( m \) with regular ordinals \( k \) and \( \lambda \) with \( \text{cf}(k) \neq \omega \neq \text{cf}(\lambda) \) and \( k \leq \lambda \), there is a model \( n \supseteq m \) with a \( k \)-scale in \( \omega_1 \) and \( c = \lambda \); and \( \exists \) models \( m \) of ZFC without \( k \)-scales for any \( k \).

**Theorem 2.** (\( \exists \) a \( k \)-scale in \( \omega_1 \)). Suppose \( \text{cf}(\lambda) = \omega \), then for \( \bigcup_\omega (\lambda + 1) \) to be paracompact it is necessary and sufficient that \( \exists \{\gamma_n : n \in \omega\} \leq \lambda \ni \sup_{n \in \omega} \gamma_n = \lambda \) and \( \bigcup_{n \in \omega} (\gamma_n + 1) \) is paracompact.

**Proof.** Necessity is obvious so we prove sufficiency.

\( \text{WLOG} \) assume \( \gamma_n < \gamma_{n+1} \) \( \forall n \in \omega \), \( \text{cf}(\gamma_n) = 1 \) \( \forall n \in \omega \), and \( \{s(\alpha) : \alpha < k\} \) is a \( k \)-scale in \( \omega_1 \) for a regular \( k \). Let \( R \) be
an open cover of $\nu^\omega(\lambda + 1)$. For each $\tau < \kappa$ and $d \in \mathcal{T}$ we construct inductively $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ to satisfy:

1. $V(d)$ and $W(d)$ are clopen subsets of $\nu^\omega(\lambda + 1)$, $\exists U \in \mathcal{U} \exists V(d) \subseteq U$, $V(d) \cup W(d) \subseteq W(d \uparrow \sigma)$ $\forall \sigma < \tau$, and if $\sigma < \tau$ is a limit ordinal $W(d \uparrow \sigma) = \bigcap_{\rho < \sigma} W(d \uparrow \rho)$.

2. If $\sigma \leq \tau$ is an odd ordinal, then
   \[
   \{V(e) : \text{dom}(e) \leq \sigma\} \cup \{W(e) : \text{dom}(e) = \sigma\}
   \]
   is a pairwise-disjoint covering of $\nu^\omega(\lambda + 1)$.

3. $A(d)$ is an infinite subset of $\omega$ and if $\sigma \leq \tau$ is a non-limit ordinal, then $A(d \uparrow \sigma) \subseteq A(d \uparrow \rho)$ $\forall \rho < \sigma$.

4. $\theta(d)(n) = \gamma_{S(\alpha)}(n) \forall n \in \omega$ and some $\alpha < \kappa$; and if $\sigma \leq \tau$ is even, then
   \[
   \{n : \theta(d \uparrow \sigma)(n) \leq \theta(d \uparrow \rho)(n)\} \equiv \phi \quad \forall \rho < \sigma.
   \]

5. If $\sigma \leq \tau$ is odd, then $W(d \uparrow \sigma) \subseteq \langle A(d \uparrow \sigma), \theta(d \uparrow \sigma) \rangle$ and
   \[
   \{V(e) : e \in \mathcal{E}, e \uparrow \sigma - 1 = \langle d \uparrow \sigma - 1 \rangle = \langle \phi, \theta(d \uparrow \sigma) \rangle \cap
   \]
   \[
   W(d \uparrow \sigma - 1).
   \]

6. If $\sigma \leq \tau$ is a non-limit even ordinal, then
   \[
   V(d \uparrow \sigma) = W(d \uparrow \sigma) \cap \langle A(d \uparrow \sigma - 1), \theta(d \uparrow \sigma - 1) \rangle.
   \]

Now suppose our objects $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ have been constructed to satisfy (1) through (6) $\forall d \in \mathcal{T}$ $\forall \tau < \kappa$.

For $x \in \nu^\omega(\lambda + 1)$ define
\[
x^\#(n) = \begin{cases} 0 & \text{if } x(n) = \lambda \\ x(n) & \text{otherwise.} \end{cases}
\]

We may find the first $\alpha \ni \{n : \gamma_{S(\alpha)}(n) \leq x^\#(n)\} = \phi$. If $\alpha = \alpha_0 + m$, where $\alpha_0 = 0$ or is a limit ordinal and $m \in \omega$, let $\tau = \alpha_0 + 2(m + 1)$. From (2), (4), (5), and (6) we have
\[
E(x) \in \bigcup \{V(e) : \text{dom}(e) \leq \tau\}.
\]

Therefore, $\{V(d) : d \in \mathcal{T}, \tau < \kappa\}$ is a pairwise-disjoint clopen refinement of $R$ covering $\nu^\omega(\lambda + 1)$. So we must complete our
construction.

Let $A(\phi) = \omega$, $W(\phi) = \gamma^\omega(\lambda + 1)$, and $\alpha$ be the first ordinal such that $E(\prod_{n \in \omega} [\gamma_{s(a)}(n), \lambda])$ is contained in some $U \in R$. Let $\theta(\phi)(n) = \gamma_{s(a)}(n) \forall n \in \omega$ and $V(\phi) = \langle A(\phi), \theta(\phi) \rangle$.

Suppose for an ordinal $\rho < \kappa$ we have constructed $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ to satisfy (1) through (6) $\forall d \in \mathbb{C} \forall \tau < \rho$. Our construction at $\rho$ needs three cases:

Case 1. $\rho$ is an odd ordinal.

Let $\tau = \rho - 1$ and $\theta(e) = \theta(d)$ if $e \in \mathbb{C}$ and $e \uparrow \tau = d$. Let

$$\{A(e) : e \in \mathbb{C}, e \uparrow \tau = d\}$$

be a listing of exactly one element from each equivalence class of elements of

$$\{A : \phi < A < A(\uparrow \sigma), \sigma \leq \tau\}.$$

For each $e \in \mathbb{C}$ we let

$$W(e) = W(e \uparrow \tau) \cap \langle A(e), \theta(e) \rangle.$$

If $d \in \mathbb{C}$, then $W(d) \cap \langle \phi, \theta(d) \rangle$ is a clopen subset of $E(\prod_{n \in \omega} [0, \theta(d)(n)])$ and $\prod_{n \in \omega} [0, \theta(d)(n)]$ is a clopen subset of a subproduct of $\prod_{n \in \omega} (\gamma_n + 1)$; therefore, by the lemma (ii) and (iii) we may find a pairwise-disjoint clopen refinement of $R, \{V(e) : e \in \mathbb{C}, e \uparrow \tau = d\}$ whose union is $W(d) \cap \langle \phi, \theta(d) \rangle$. Clearly, (1) through (6) are satisfied.

Case 2. $\rho$ is a non-limit even ordinal.

Let $\tau = \rho - 1$, and $A(e) = A(d)$, and $W(e) = W(d)$ if $e \in \mathbb{C}$ and $e \uparrow \tau = d$. If $d \in \mathbb{C}$ and $W(d) = \phi$, we let $W(e) = V(e) = \phi$ and

$$\theta(e)(n) = \gamma_{s(a)}(n) \forall n \in \omega; \text{ where}$$

$$\theta(e \uparrow \tau)(n) = \gamma_{s(a)}(n) \forall n \in \omega.$$
If \( d \in T^c \), \( W(d) \neq \emptyset \), and

\[
Y(d) = \{ g \upharpoonright \omega - A(d) : g^{-1}(\lambda) = A(d), E(g) \in W(d) \} = \emptyset.
\]

In \( \cap_{n \in A(d)} (\theta(d)(n) + 1) \), let

\[
R(d) = \{ E(\prod_{n \notin A(d)} U(n)) : E(\prod_{n \in \omega} U(n)) \subseteq \text{some } U \in R, \exists E(g) \in W(d) \cap E(\prod_{n \in \omega} U(n)), g^{-1}(\lambda) = A(d) \}
\]

Since \( \cap_{n \in \omega} (\gamma_n + 1) \) is homeomorphic to a clopen subset of a subproduct of \( \prod_{n \in \omega} (\gamma_n + 1) \), we may use the lemma, (ii) and (iii), to find a pairwise disjoint clopen refinement \( \{ G(\delta) : \delta < \xi \} \) of \( R(d) \) whose union is \( E(Y(d)) \). If \( e \in \emptyset^c \), \( e \upharpoonright \tau = d \), \( e(\tau) = \delta \), then let \( \alpha(\delta) \) be the first ordinal > \( \alpha(d) \), where \( \theta(d)(n) = Y_S(\alpha(d))(n) \forall n \in \omega \), such that

\[
V(e) = \{ E(x) \in W(d) : E(x \upharpoonright \omega - A(d)) \subseteq G(\delta), x(n) > Y_S(\alpha(\delta))(n) \forall n \in \omega \}
\]

is contained in a member of \( R \). Let \( \theta(e)(n) = Y_S(\alpha(\delta))(n) \forall n \in \omega \).

Clearly, (1) through (6) are satisfied.

**Case 3.** \( \rho \) is a limit ordinal.

If \( e \in \emptyset^c \), let \( A(e) = \omega \), \( V(e) = \emptyset \), and \( \theta(e)(n) = Y_S(\alpha)(n) \forall n \in \omega \), where

\[
\alpha = \sup \{ \beta : \theta(e \upharpoonright \tau) \in W(\beta)(n) \forall n \in \omega, \tau < \rho \}.
\]

To see that (1) through (6) are satisfied, we must show

\[
W(e) = \cap_{\tau < \rho} W(e \upharpoonright \tau) \text{ is open.}
\]

However, if \( E(x) \in W(e) \), then the induction hypothesis and the definition of \( W(d) \) in Case 2 yields

\[
E(\prod [x^*(n), x(n)]) \subseteq W(e),
\]

where

\[
x(n) \text{ if } \text{cf}(x(n)) = 1
\]

\[
x^*(n) = -\theta(e)(n + 1) \text{ if } x(n) \text{ is a limit > } \theta(e)(n)
\]

\[
\sup \{ \theta(e \upharpoonright \tau) : \theta(e \upharpoonright \tau)(n) < x(n), \tau < \rho \} + 1, \text{ otherwise.}
\]
This completes the construction and the proof of Theorem 2.

Remarks

A. There are many models of ZFC, constructed via forcing, in which there are no $k$-scales [2]. However, J. Roitman [4] has shown that in some of these models, techniques inadvertedly, in some sense, yield $\sqcup_{n \in \omega} X_n$ paracompact $\forall$ compact metrizable $X_n$; specifically she has shown:

In a model $m$ of set theory which is a direct iterated CCC extension of length $k$ of a model $n$, $\text{cf}(k) > \omega \Rightarrow \sqcup_{n \in \omega} X_n$ is paracompact if $X_n$ is regular and separable. A simple adaptation of her proofs will give the conclusion of Theorem 2 in $m$.

B. Suppose $\omega_0$ is an ordinal and for $n > 0$ $\omega_n$ is the lexicographic ordered product of $\omega_{n-1}$ with itself. Let $\omega = \sup_{n \in \omega} \omega_n$. It is unknown whether (exists a c-scale in $\omega$) $\Rightarrow \sqcup_{n \in \omega} (\omega_0 + 1)$ is paracompact when $\omega_0 = \omega_1$; however, our theorems show (exists a $k$-scale in $\omega$) $\Rightarrow \sqcup_{n \in \omega} (\omega_n + 1)$ is paracompact $\forall n \in \omega$. It is unknown whether $\sqcup_{n \in \omega} (\omega + 1)$ is paracompact $\Rightarrow \sqcup_{n \in \omega} (\omega_n + 1)$ is paracompact when $\omega_0 = \omega$; although $\sqcup_{n \in \omega} (\omega_n + 1)$ is paracompact for each $n$. The simplest question still unanswered is "Does there exist a model $m$ of ZFC in which $\sqcup_{n \in \omega} (\lambda + 1)$ is not paracompact for some ordinal $\lambda".$ The hardest question asks that $\lambda = \omega$.

C. We observe a recent result communicated to the author by E. K. van Douwen: If $X_n$ is compact $\forall n \in \omega$, then $\sqcup_{n \in \omega} X_n$ is pseudo-normal. The author gives much appreciation to the referee whose suggestions for clarification of

$\sqcup_{n \in \omega} (\omega_n + 1)$ may be embedded in $\sqcup_{n \in \omega} (\omega_n + 1)$.\[4\]
unnecessary technicalities in our proofs appear.

Added in proof

Recently, J. Roitman has proved that $\bigsqcup_{n \in \omega} X_n$ is paracompact whenever each $X_n$ is compact first countable and $\omega \omega$ fails to have a cofinal family of cardinality less than the continuum. A corollary to this theorem and our theorems 1 and 2 yields $c = \omega_2 \Rightarrow \bigsqcup^\omega \omega_1 + 1$ is paracompact. Independently, I have shown the same corollary and, in addition:

Suppose, in theorem 2, $(\exists \kappa$-scale in $\omega \omega)$ is replaced by $\kappa$ is the least cardinal of any cofinal family in $\omega \omega$ and $A \subseteq \mathcal{P}(\omega)$ with $|A| = \kappa$, then

$$E(\{x \in \bigsqcup \omega \lambda + 1: x^{-1}(\lambda) \in A\})$$

is paracompact.

References

1. E. K. van Douvan, $\exists$ a $\kappa$-scale implies $\bigsqcup_{n \in \omega} X_n$ is paracompact if $X_n$ is compact metric $\forall n \in \omega$, lecture presented at the Ohio University Conference on Topology, May 1976.
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