A NOTE ON THE PRODUCT OF
FRECHET SPACES

by

GARY GRUENHAGE
A NOTE ON THE PRODUCT OF FRECHET SPACES

Gary Gruenhage

1. Introduction

A space $X$ is said to be a Frechét space if whenever $x \in A$, there exist $x_n \in A$, $n = 1, 2, \ldots$, with $x_n \to x$. In general, Frechét spaces behave very badly with respect to products. In fact, if $X$ and $Y$ are non-discrete Frechét spaces and $X \times Y$ is Frechét, then a theorem of Michael [5] implies that $X$ and $Y$ must have the following stronger property: if $x \in \bigcap_{n=1}^{\infty} A_n$, where $A_1 \supset A_2 \supset \ldots$, then there exists $x_n \in A_n$ with $x_n \to x$. Spaces satisfying this property are called countably bi-sequential spaces. We should add that even if $X$ and $Y$ are countably bi-sequential, this does not guarantee that $X \times Y$ is Frechét (see [4] or [6]).

In a letter to the author, F. Galvin asked the following question: if $X_0, X_1, X_2, \ldots$ are such that $\prod_{i \leq n} X_i$ is Frechét (equivalently, countably bisequential) for all $n \in \omega$, must $\prod_{i \in \omega} X_i$ be Frechét (equivalently, countably bi-sequential)? Y. Tanaka [8, Problem 2.6] has asked the same question. In this paper, we construct, assuming Martin's Axiom (MA), a Frechét space $X$ such that $X^n$ is Frechét for all $n \in \omega$, but $X^\omega$ is not Frechét. The space $X$ is countable, and has only one non-isolated point.

Before proceeding with the construction of the example, we would like to mention some related problems. Bi-sequential spaces [5] are closed under countable products, so the space $X$ we construct is a countable countably bi-sequential space.
which is not bi-sequential. Others (e.g., Galvin [2], Malyhin [4], Olson [6]) have constructed such spaces assuming various axioms of set theory, but no real example has been found. (There are uncountable real examples, e.g., an uncountable \( \mathbb{I} \)-product of the unit interval.) A space \( X \) is called a \( w \)-space if whenever \( x \in A_n \), \( n = 1, 2, \ldots \), there exists \( x_n \in A_n \) with \( x_n \to x \). These spaces were introduced by the author in [3], and defined in terms of an infinite game, but this characterization, due to P. L. Sharma [7], is much better. Clearly, every \( w \)-space is countably bi-sequential, and the difference between the two classes of spaces does not, on the surface, look very large. But the following question, also asked by Galvin, remains open: if \( X_n \) is a \( w \)-space for all \( n \in \omega \), must \( X^\omega \) be a \( w \)-space (or a Fréchet space)? A counterexample to this question would be about as far as one could go in this direction. Call \( X \) a \( c^* \)-space (terminology due to Sharma) if \( X \) has countable tightness and every countable subset of \( X \) is first countable. It is easy to see that if \( X^n \) is a \( c^* \)-space for every \( n \in \omega \), then \( X \) is a \( c^* \)-space. No real example of a space which is a \( w \)-space but not a \( c^* \)-space has been found. However, Galvin [1] has constructed such spaces assuming MA.

2. Construction of the Example

Unless otherwise stated, we use the letters \( m, n, \) and \( k \) to denote natural numbers. The example is based on a construction, by induction on the ordinals less than the continuum \( c \), of a certain collection of almost-disjoint subsets of \( \omega \). To get us past an uncountable stage \( \alpha < c \), we need the
following lemma:

Lemma (MA). Let \( \{ I_\alpha \}_{\alpha < \kappa} \) be a collection of infinite almost-disjoint subsets of \( \omega \). Suppose \( A = \omega^n \times \omega^m \), and \( \{ a(0), a(1), \ldots, a(m-1) \} \subset \kappa \) are such that

1. \( A \subset \omega^n \times \prod_{j<m} I_{a(j)} \)
2. \( A \cap (\prod_{i<n} (\omega \setminus E(i)) \times (\prod_{j<m} I_{a(j)} \setminus F(j))) \neq \emptyset \) whenever
   \[ E(i) \text{ is a finite union of the } I_{a} 's, \text{ together with a finite subset of } \omega, \text{ and } F(j) \text{ is a finite subset of } \omega. \]

Then there is a sequence \( \tilde{x}_0, \tilde{x}_1, \ldots \) of elements of \( A \) such that

(i) \( C(\tilde{x}_i) \cap C(\tilde{x}_j) = \emptyset \) whenever \( i \neq j \), where \( C(\tilde{x}) \) is the set of coordinates of \( \tilde{x} \);

(ii) if \( \alpha < \kappa \), then \( I_{\alpha} \cap \{ \pi_i(\tilde{x}_j): i < n, j \in \omega \} \) is finite, where \( \pi_i \) is the projection on the \( i \)th coordinate.

Proof. Let \( P = \{(f,F): f \subset A, F \subset \kappa, \text{ with } f \text{ and } F \text{ finite}\} \). Define \( (f,F) < (g,G) \) if and only if

(a) \( f \subset g \) and \( F \subset G \);

(b) if \( \tilde{y} \in g \setminus f \), then \( \tilde{y} \) is an element of \( A \cap (\bigcup_{i<n} (\bigcup_{\alpha \in F} I_{\alpha}) \cup (\bigcup_{x \in f} C(\tilde{x})) \times (\prod_{j<m} I_{a(j)} \setminus \bigcup_{x \in f} C(\tilde{x})) \). So defined, \( (P,<) \) satisfies the CCC because there are only countably many possible \( f \) 's, and \( (f,F) \) and \( (f,G) \) are bounded by \( (f,F \cup G) \). For each \( \alpha < \kappa \), and \( i \in \omega \) let \( X_{\alpha,i} = \{(f,F) \in P: |f| > i \text{ and } \alpha \in F\} \). \( X_{\alpha,i} \) is a dense open set in \( (P,<) \), so by MA, there is a compatible family \( \{(f_{\alpha,i}, F_{\alpha,i}) \in X_{\alpha,i}: \alpha < \kappa, i \in \omega\} \). Pick \( \tilde{x}_0 \in f_{\alpha(0),i(0)} \). If \( \tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{k-1} \) have been chosen, pick \( \tilde{x}_k \in f_{\alpha(k),i(k)} \setminus \bigcup_{j<k} f_{\alpha(j),i(j)} \). We claim that \( \tilde{x}_0, \tilde{x}_1, \ldots, \) is the desired sequence. If \( j < k \), then since \( \tilde{x}_k \in f_{\alpha(k),i(k)} \setminus \bigcup_{j<k} f_{\alpha(j),i(j)} \), and by the compatibility,
the conclusion of property (b) is satisfied with $\hat{y} = \hat{x}_k$ and $f = f_{a(j),i(j)}$. Hence $C(\hat{x}_j) \cap C(\hat{x}_k) = \emptyset$, and so property (i) of the conclusion of the lemma is satisfied. Now let $\alpha < \kappa$. If $\hat{x}_k \notin f_{a,1}$, then the conclusion of (b) is satisfied with $\hat{y} = \hat{x}_k$ and $F = F_{a,1}$. Since $\alpha \in F_{a,1}$, the first $n$ coordinates of $\hat{x}_k$ miss $I_\alpha$. Thus (ii) is satisfied, and this completes the proof.

**Theorem (MA).** There is a countable Fréchet space $X$ such that $X^n$ is Fréchet for all $n \in \omega$, but $X^\omega$ is not Fréchet.

**Proof.** We will construct a countable space $X_k$ for each $k \in \omega$, so that $\prod X_k$ is Fréchet for all $n \in \omega$, but $\prod_{k < n} X_k$ is not Fréchet. We can then take $X$ to be the free union of the $X_k$'s.

To this end, we will construct a sequence $\{\mathcal{I}_n\}_{n \in \omega}$ of collections of infinite subsets of $\omega$ such that $\bigcup_{n \in \omega} \mathcal{I}_n$ is a maximal almost-disjoint collection. We then take $X_k$ to be the space $\omega \cup \{\omega\}$ with the points of $\omega$ isolated, and a neighborhood of $\omega$ is $\omega \cup \{\omega\}$ minus a finite union of elements of $\bigcup_{n \in \omega} \mathcal{I}_j$. It is easy to see that, in the space $\prod X_k$, the point $(\omega, \omega, \ldots) \in \text{Cl}\{(n, n, \ldots): n \in \omega\}$, but no sequence of the type $\{(n_k, n_k, \ldots): k \in \omega\}$ converges to $(\omega, \omega, \ldots)$. Thus $\prod X_k$ is not a Fréchet space.

We need to construct the $\mathcal{I}_k$'s so that every finite product of the $X_k$'s is Fréchet. First construct $I_k(n)$, $n \in \omega$, so that $\{I_k(n): n \in \omega, k \in \omega\}$ is an almost-disjoint collection of infinite subsets of $\omega$, with the additional property that for each $k \in \omega$ and finite subset $F$ of $\omega$, there is $n \in \omega$ with $F \subseteq I_k(n)$. 
For each \( n \in \omega \), let \( A_n = P(\omega^n) \), and let \( A = \bigcup_{n \in \omega} A_n \). Let \( A = \{ A_\alpha : \alpha < c \} \) so that each element of \( A \) appears \( c \) times in the well-ordering. For each \( n \in \omega \), define \( \beta(n) = n \). Now suppose \( I_k(a) \) and \( \beta(a) \) have been defined for all \( \alpha < \kappa \), where \( \omega < \kappa < c \), and \( k \in \omega \). Let \( J(\kappa) = \{ I_k(a) : \alpha < \kappa, k \in \omega \} \). Let \( \beta(\kappa) \) be the least ordinal \( \beta \) such that \( \beta > \beta(a) \) whenever \( \omega < \alpha < \kappa \), and such that \( A_\beta \subseteq \omega^n \) satisfies the following two properties:

(i) there are a set \( J \subseteq \{0,1,\ldots,n-1\} = n \), and \( \{ I_j : j \in J \} \subseteq J(\kappa) \) so that \( A_\beta \subseteq (\prod_{i \in n \setminus J} \omega) \times (\prod_{j \in J} I_j) \);

(ii) \( A_\beta \cap [(\prod_{i \in n \setminus J} \omega \setminus E(i)) \times (\prod_{j \in J} I_j \setminus F(j))] \neq \emptyset \) whenever \( E(i) \) is a finite union of elements of \( J(\kappa) \), and \( F(j) \) is a finite subset of \( \omega \).

Note that \( n \) is uniquely determined by \( A_\beta \), but the set \( J \) depends also on \( \kappa \). Also, such a \( \beta \) always exists since \( \omega \) itself, with \( n = 1 \) and \( J = \emptyset \), satisfies (i) and (ii).

By the lemma, there is a sequence \( x_0, x_1, \ldots \) in \( A_{\beta}(\kappa) \) such that \( C(x_i) \cap C(x_j) = \emptyset \) for \( i \neq j \), and \( I \cap \{ \pi_i(x_k) : k \in \omega, i \in n \setminus J \} \) is finite whenever \( I \in J(\kappa) \). Express \( \omega \) as \( \bigcup_{m \in \omega} W_m \), where \( W_m \) is infinite and \( W_m \cap W_{m'} = \emptyset \) if \( m \neq m' \). Define \( I_m(\kappa) = \{ \pi_i(x_k) : k \in W_m, i \in n \setminus J \} \). The inductive step is now complete.

Let \( J(\kappa) = \{ I_k(a) : \alpha < c \} \), and let \( X_k \) be as defined earlier. We have already shown that \( \prod X_k \) is not Fréchet. It remains to prove that \( \prod X_k \) is Fréchet for each \( n \in \omega \). To this end, suppose \( A \subseteq \prod X_k \), and \( x \in \bigcap A \). We need to show there exists \( x_n \in A \) with \( x_n \to x \). We will prove this for the case \( A \subseteq \omega^n \) and \( x = (\omega, \omega, \ldots, \omega) = \omega^n \), the other cases being trivial or reducible to a case similar to this one.
Let $\mathcal{G} = \bigcup_n \mathcal{G}_n$. Suppose $A \cap (\prod_{i<n} \omega \setminus E(i)) = \emptyset$, where $E(i)$ is a finite union of elements of $\mathcal{G}$. Then $A \subseteq \bigcup_{i<n} (\omega \times \cdots \times \omega \times E(i) \times \omega \times \cdots \times \omega)$, so there exists $j(0) < n$ and $I_j(0) \in \mathcal{G}$ so that $I_j(0) \subseteq E(j(0))$, and $\omega^n \in \text{Cl}(A(0))$, where $A(0) = A\cap[\omega \times \cdots \times \omega \times I_j(0) \times \omega \times \cdots \times \omega] = A(0) \cap \prod_{i<n} \omega \times I_j(0) \times \omega \times \cdots \times \omega].$ Now suppose $A(0) \cap \{(\prod_{i\in n\setminus\{j(0)\}} \omega \setminus E(i)) \times (I_j(0) \setminus D(j(0))) = \emptyset$, where $E(i)'$ is a finite union of elements of $\mathcal{G}$ and $D(j(0))$ is a finite subset of $\omega$. (We are using the subscript to indicate position in the product, in order to simplify notation.) Then there exists $j(1) \in n\setminus\{j(0)\}$ so that $\omega^n \in \text{Cl}(A(1))$, where $A(1) = A(0) \cap [\omega \times \cdots \times \omega \times I_j(0) \times \omega \times \cdots \times \omega \times I_j(0) \times \omega \times \cdots \times \omega] = A(0) \cap \prod_{i<n\setminus\{j(0), j(1)\}} \omega \times I_j(0) \times \omega \times \cdots \times \omega].$ We continue the process until we have a set $J = \{j(0), \cdots, j(m)\}$ and $A(m) \subseteq (\prod_{i \in n} \omega) \times \prod_{j \in J} I_j$ with $\omega^n \in \text{Cl}(A(m))$ and $A(m) \cap \{(\prod_{i \in n\setminus J \setminus \{j(0), \cdots, j(m)\}} \omega \setminus E(i)) \times (\prod_{j \in J \setminus \{j(0), \cdots, j(m)\}} I_j \setminus F(j)) \neq \emptyset$ whenever $E(i)$ is a finite union of elements of $\mathcal{G}$ and $F(j)$ is a finite subset of $\omega$.

Choose $\kappa_0$ large enough so that $\{I_j : j \in J\} \subseteq \mathcal{G}(\kappa_0)$. Now $A(m) = A_{\beta_0}$ for $c$ $\beta$'s, so choose $\beta_0 > \sup \{\beta(\alpha) : \alpha < \kappa_0\}$ such that $A(m) = A_{\beta_0}$. Then for any $\kappa_0 < \kappa < c$, it is true that $A_{\beta_0} \cap J$, and $\kappa$ satisfy (i) and (ii) in the above construction of the $\mathcal{G}_k$'s. Thus $\beta_0 = \beta(\kappa)$ for some $\kappa_0 < \kappa < c$, and we have the sequence $\vec{x}_0, \vec{x}_1, \cdots$ in $A_{\beta(\kappa)}$ that we chose in the construction. It is easy to see from the definition of $X_i$ that the set $\{\pi_i(\vec{x}_k) : k \in W_n\}$ converges to $\omega$ in $X_i$ for each $i < n$, and since $C(X_j) \cap C(\vec{x}_k) = \emptyset$ for $j \neq k$, then $\{\vec{x}_k : k \in W_n\}$ converges to $\omega^n$. This completes the proof.

Remark. We can get an example with only one non-isolated
point as follows: let Y be the space which is the free union $X_k$'s, with the points "∞" identified to a single point $\hat{\infty}$. Let $\pi: X + Y$ be the projection. Define a neighborhood of $\hat{\infty}$ to be of the form $\pi(U_1 \cup \cdots \cup U_n \cup X_{n+1} \cup X_{n+2} \cup \cdots)$, where $U_i$ is an open set in $X_i$ containing $\infty$.

References


Auburn University

Auburn, Alabama 36830