ON THE CHARACTER OF
SUPERCOMPACT SPACES

by

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1. Introduction, Definitions and Conventions

A collection of subsets \( J \) of a space \( X \) is called a \( \pi \)-network for \( x \in X \) provided that every neighborhood of \( x \) contains a member from \( J \). The supertightness \( p(x,X) \) of \( x \) in \( X \) is defined to be the least cardinal \( \kappa \) for which every \( \pi \)-network \( J \) for \( x \) consisting of finite subsets of \( X \) contains a subfamily \( J' \subseteq J \) of cardinality \( \leq \kappa \) which is a \( \pi \)-network for \( x \). In addition, the supertightness \( p(X) \) of \( X \) is defined by

\[
p(X) = \omega \cdot \text{sup} \{ p(x,X) | x \in X \}.
\]

It is clear that \( t(X) \leq p(X) \) for every topological space \( X \) (for the definitions of cardinal functions such as \( t,w,d,c,\chi \) see Juhász [7]); in addition the reader can easily verify that \( p(X) = t(X,H_f(X)) \), where \( H_f(X) \) denotes the hyperspace of finite nonempty subsets of \( X \).

For every compact Hausdorff space \( X \) and \( k \in \omega \) we say that \( \text{cmpn}(X) \leq k \) provided that there is an open subbase \( U \) for \( X \) such that every covering of \( X \) by elements of \( U \) contains a subcovering consisting of at most \( k \) elements of \( U \). In addition, \( \text{cmpn}(X) = k \) if \( \text{cmpn}(X) \leq k \) and \( \text{cmpn}(X) \neq k \) and \( \text{cmpn}(X) = \infty \) in case \( \text{cmpn}(X) \neq k \) for all \( k \in \omega \). \( \text{cmpn}(X) \) is called the compactness number of \( X \) (cf. Bell \& van Mill [3]). It is known that for every \( k \in \omega \) there is a compact Hausdorff space \( X_k \) for which \( \text{cmpn}(X_k) = k \); also \( \text{cmpn}(\beta \omega) = \infty \) (cf. Bell

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& van Mill [3]). Spaces with compactness number less than or
equal to 2 are just the supercompact spaces as defined by
de Groot in [6]. Many spaces are supercompact, for example
all compact metric spaces (cf. Strok & Szymański [14]; ele­
mentary proofs of this fact have recently been discovered by
van Douwen [4] and Mills [12]). The first examples of non­
supercompact compact Hausdorff spaces were found by Bell [1].

In section 2 of the present paper we will prove a theorem
from which the following statement is a corollary:

If X is supercompact then \( \chi(X) \leq d(X) \cdot p(X) \).

The supercompactness of X is essential; we will give an
example of a space X such that cmpn(X) = 3, d(X) = p(X) = \( \omega \)
and \( \chi(X) = 2^{\omega} \). In addition we show that the inequality can­
not be sharpened by considering \( t \) instead of \( p \). We construct
an example of a supercompact space X such that \( d(X) = t(X) = \omega \)
while \( \chi(X) = p(X) = 2^{\omega} \).

We are indebted to Eric van Douwen for some helpful com­
ments.

2. On the Character of Supercompact Hausdorff Spaces

All topological spaces under discussion are assumed to
be Tychonoff.

Let X be a set and let \( \kappa \) be a cardinal. We define (as
usual)

\[
[X]^{\kappa} = \{ A \subseteq X \mid |A| = \kappa \}
\]

\[
[X]^{<\kappa} = \{ A \subseteq X \mid |A| < \kappa \}
\]

\[
[X]^{\leq\kappa} = \{ A \subseteq X \mid |A| \leq \kappa \}.
\]

Let X be a space, B be a closed subset of X, and Y be
the space obtained from X by identifying B to one point. Let
f: X → Y be the identification. For φ ∈ {t, p, χ} let

φ(B, X): = φ(f[B], Y).

In case X is supercompact, the supercompactness of X
can also be described in terms of a closed subbase: a space
is supercompact iff it has a closed subbase with the property
that any of its linked (= every two of its members meet) sub­
collections has nonvoid intersection. Such a subbase is
called binary. Without loss of generality we may assume that
a binary subbase is closed under arbitrary intersections.
Let S be a binary subbase for X. For A ⊆ X define I(A) ⊆ X
by

I(A): = ∩{S ∈ S|A ⊆ S}.

Notice that cl_X(A) ⊆ I(A), since each element of S is closed,
that I(I(A)) = I(A) and that I(A) ⊆ I(B) if A ⊆ B ⊆ X. The
following lemma was proved in van Douwen & van Mill [5].
For the sake of completeness we will give its proof also here.

2.1. Lemma (van Douwen & van Mill [5]). Let S be a
binary subbase for X and let p ∈ X. If U is a neighborhood
of p and if A is a subset of X with p ∈ cl_X(A), then there is
a subset B of A with p ∈ cl_X(B) and I(B) ⊆ U.

Proof. Since X is regular, p has a neighborhood V such
that p ∈ cl_X(V) ⊆ U. Let J be the collection of all finite
intersections of elements of S. Choose a finite J ⊆ J such
that cl_X(V) ⊆ ∪J ⊆ U. Now J is finite, and A ∩ V ⊆ ∪J, and
p ∈ cl_X(A ∩ V); hence there is an S ∈ J with p ∈ cl_X(A ∩ V
∩ S). Let B: = A ∩ V ∩ S. Then p ∈ cl_X(B), and B ⊆ A, and
I(B) ⊆ S ⊆ ∪J ⊆ U.

We now can prove the main result of this section.
2.2. Theorem. Let $Y$ be a continuous image of a super-compact space. Then $\chi(Y) \leq d(Y) \cdot p(Y)$.

Proof. Let $S$ be a binary subbase for $X$ which is closed under arbitrary intersections and let $f: X \to Y$ be a continuous surjection. Let $\kappa = d(Y) \cdot p(Y)$ and fix a dense subset $D = \{d_\alpha | \alpha < \kappa\}$ of $Y$. Choose $y \in Y$ and define

$$J : = \{ \bigcup_{g | g \in [S]^{< \omega}} \text{ and } \exists \text{ neighborhood } U \text{ of } y \text{ such that } f^{-1}(U) \subset \bigcup_{g} \}.$$ 

Notice that for every neighborhood $U$ of $y$ there is an $F \in J$ such that $f^{-1}(y) \subset F \subset f^{-1}(U)$ since $S$ is a subbase. For each $F \in J$ let $L = \bigcup_{i \leq n(F)} S^F_i$, where $S^F_i \in S$ for all $i \leq n(F)$. For each $\alpha < \kappa$ take $d'_\alpha \in X$ such that $f(d'_\alpha) = d_\alpha$.

Fix $\alpha < \kappa$ and $F = \bigcup_{i \leq n(F)} S^F_i \in J$. For each $i \leq n(F)$ pick a point $e_\alpha^i \in \bigcap_{S \in S^F_i} \bigcap \{d'_\alpha, s\} \cap S^F_i$. Notice that, since $S$ is binary, it is possible to take such a point. Let $E^\alpha(F) = \{e_0^\alpha, \ldots, e_n^\alpha(F)\}$. Then $\{f(E^\alpha(F)) | F \in J\}$ is a collection of finite subsets of $Y$ such that each neighborhood of $y$ contains a member of it. Since $p(y,Y) \leq \kappa$ we can find a subfamily $J_\alpha \subset J$ of cardinality at most $\kappa$ such that each neighborhood of $y$ contains a member of $\{f(E^\alpha(F)) | F \in J_\alpha\}$.

We claim that

$$\bigcap_{\alpha \leq \kappa} J_\alpha \cap \overline{\text{cl}_X \{d'_\alpha | \alpha < \kappa\}} = f^{-1}(y) \cap \overline{\text{cl}_X \{d'_\alpha | \alpha < \kappa\}}$$

which proves that $\chi(y,Y) \leq \kappa$ since $|\bigcap_{\alpha \leq \kappa} J_\alpha| \leq \kappa \cdot \kappa = \kappa$. To this end, first observe that $f^{-1}(y) \subset \bigcap_{\alpha \leq \kappa} J_\alpha$. Assume that $(*)$ is not true; then there is an $x \in (\bigcap_{\alpha \leq \kappa} J_\alpha) \cap \overline{\text{cl}_X \{d'_\alpha | \alpha < \kappa\}} - (f^{-1}(y) \cap \overline{\text{cl}_X \{d'_\alpha | \alpha < \kappa\}})$. Then clearly $f(x) \neq y$.
and consequently we may take disjoint neighborhoods \( U \) and \( V \) of, respectively, \( y \) and \( f(x) \). By lemma 2.1 we can find a subset \( D'_0 = \{d'_a | a < \kappa\} \) such that \( x \in I(D'_0) \subset f^{-1}(V) \). Pick \( d'_{a_0} \in D'_0 \) arbitrarily. In addition, take \( F \in J_{a_0} \) such that 
\[
E_{a_0}(F) \subset f^{-1}(U).
\]
Since \( x \in \cap_{a < \kappa} J_a \) we have that \( x \in F = \bigcup_{i \in \mathbb{N}(F)} S_i^F \); hence there is an \( i_0 \leq \mathbb{N}(F) \) such that \( x \in S_{i_0}^F \).

Then \( e_{i_0}^{a_0} \in \cap_{s \in S_{i_0}} I(\{d'_{a_0}, s\}) \cap S_{i_0}^F \subset I(\{d'_{a_0}, x\}) \cap S_{i_0}^F \subset I(D'_0) \cap S_{i_0}^F \subset f^{-1}(V) \). This is a contradiction, however, since 
\[
e_{i_0}^{a_0} \in f^{-1}(U) \text{ and } f^{-1}(U) \cap f^{-1}(V) = \emptyset.
\]

2.3. Corollary. Let \( X \) be a supercompact space and let \( B \) be a closed subset of \( X \). Then \( \chi(B) \leq d(X) \cdot p(B, X) \).

We will now describe the examples announced in the introduction. We start with a useful result, the proof of which was suggested to us by Eric van Douwen. Our original proof was much more complicated.

2.4. Theorem. Let \( \gamma X \) be a compactification of a separable metric space \( X \) such that \( \gamma X - X \) is homeomorphic to the one point compactification of a discrete space. Then \( p(\gamma X) = \omega \).

Proof. Write \( \gamma X - X \) as \( D \cup \{\infty\} \), where \( \infty \) is the non-isolated point. Evidently \( p(x, \gamma X) = \omega \) for all \( x \neq \infty \). It remains to show that \( p(x, \gamma X) = \omega \). Let \( B \) be a countable base for \( X \) closed under finite union.

For \( A, C \subseteq \mathcal{P}(\gamma X) \) and \( S \subseteq \gamma X \) we say that \( C \) covers \( A(\text{rel } S) \) if for every neighborhood \( U \) of \( \infty \) with \( U \supseteq S \) the following holds: if there is there is \( A \in A \) with \( A \subseteq U \) then there is
C ∈ C with C ⊆ U. We say that C covers A if C covers A(\text{rel } \emptyset).

We prove that \( p(\omega, \gamma X) = \omega \) by proving something formally stronger:

(1) for all \( J \subseteq [\gamma X]^{<\omega} \) there is \( J' \in [J]^{<\omega} \) which covers \( J \).

So let \( J \subseteq [\gamma X]^{<\omega} \). For \( B \in \beta \) and \( n \in \omega \) define

\[
J_{B,n} = \{ F \in J : F \cap X \subseteq B, |F \cap D| = n \}.
\]

[We do not care if \( \omega \in F \) or not.] Using the fact that \( \beta \) is closed under finite unions, one can easily prove that (1) follows from

(2) for all \( B \in \beta \) and \( n \in \omega \) there is \( J'_{B,n} \in [J_{B,n}]^{<\omega} \) which covers \( J_{B,n} \) (\text{rel } B).

But evidently (2) follows from

(3) for all \( n \in \omega \), if \( A \subseteq [D]^n \) then there is \( A' \in [A]^{<\omega} \) which covers \( A \).

We prove (3) with induction on \( n \). For \( n = 0 \) there is nothing to prove. Suppose (3) holds for a certain \( n \in \omega \), and let \( A \subseteq [D]^{n+1} \). Let \( M \) be a maximal disjoint subfamily. If \( M \) is infinite let \( A' \) be any member of \( [M]^{\omega} \). If \( M \) is finite

\[
A_x = \{ A \in A : x \in A \} \quad (x \in \cup M)
\]

For each \( x \in \cup M \) there is \( A'_x \in [A_x]^{<\omega} \) which covers \( A_x \). Now let \( A' = \cup_{x \in \cup M} A' \).

This theorem gives us our first example.

2.5. Example. A compact space \( X \) such that \( \text{cmpn}(X) = 3 \), \( d(X) = p(X) = \omega \) while \( \chi(X) = 2^{\omega} \).

Indeed, let \( X \) be the one point compactification of the Cantor tree \( {}^\omega 2 \cup {}^\omega 2 \) (cf. Rudin [13]). In van Douwen &
van Mill [5] it was shown that this space has compactness number 3 (this was also shown independently by M. G. Bell). Theorem 2.5 gives us \( p(X) = \omega \) while clearly \( d(X) = \omega \) and \( \chi(X) = 2^\omega \).

We will now describe our second example.

2.6. Example. A supercompact space \( Z \) for which \( d(Z) = t(Z) = \omega \) and \( \chi(Z) = 2^\omega \).

Indeed, let \( L \) be the "double arrow line," i.e. the space \([0,1] \times 2\) lexicographically ordered. Let \( A \subset L^2 \) be the set \( \{(x,y) | y \geq x \} \). Then set \( Z = L^2 / A \), and let \( \pi : L^2 \rightarrow X \) be the projection. Since \( L \) is first countable, so is \( L^2 \); we conclude that \( t(L^2) = \omega \). This implies that \( t(Z) = \omega \) since \( \pi \) is closed. Clearly \( d(Z) = \omega \). Since \( L^2 - A \) contains \( \{(a,1),(a,0)\} | a \in [0,1] \} \) as a closed discrete subset of cardinality \( 2^\omega \), \( A \) is not a \( G_\delta \) in \( L^2 \) so that \( \chi(Z) > \omega \). In fact, it is easily seen that \( \chi(Z) = 2^\omega \). It remains only to show that \( X \) is supercompact.

To this end, let \( A_0 \) be the set of all clopen rectangles in \( L^2 \) which do not meet \( A \) (a rectangle is the product of two intervals). In addition, let \( A_1 : = \{[a,b]^2 | [a,b] \text{ is clopen in } L \} \). It is easily verified that \( \{\pi[B] | B \in A_0 \cup A_1 \} \) is a binary closed subbase for \( Z \).

The above space \( Z \) of example 2.7 has another surprising property; it is the continuous image of a normally supercompact space while \( \chi(Z) < d(Z) \cdot t(Z) \). Below we will prove that for every normally supercompact space \( X \) the inequality \( \chi(X) \leq d(X) \cdot t(X) \) holds. Hence, in contrast with Theorem 2.2,
this is not true for continuous images of normally super-
compact spaces.

Recall that a normally supercompact space is a space $X$
which possesses a binary subbase $S$ which in addition is normal,
i.e. for all disjoint $S_0, S_1 \in S$ there are $T_0, T_1 \in S$ such that
$S_0 \subseteq T_0 - T_1, S_1 \subseteq T_1 - T_0$ and $T_0 \cup T_1 = X$. This is not such
a strange condition, since in van Mill & Schrijver [10] it
was shown that if $S$ is a binary subbase for $X$ then $S$ is
weakly normal, i.e. for all disjoint $S_0, S_1 \in S$ there is a
finite covering $\mathcal{M}$ of $X$ by elements of $S$ such that each ele-
ment of $\mathcal{M}$ meets at most one of $S_0$ and $S_1$. However, the
normally supercompact spaces have much stronger properties
than the supercompact spaces, see van Mill [9]. We also
want to notice that there is a geometric characterization of
normally supercompact spaces, see van Mill & Wattel [11].

Since it is easily seen that each product of linearly
orderable compact spaces is normally supercompact we see that
the space $Z$ of example 2.6 is the continuous image of a
normally supercompact space.

2.7. Lemma. Let $S$ be a binary normal subbase for $X,$
let $x \in X$ and let $U$ be a neighborhood of $x$. Then there is
a neighborhood $V$ of $x$ such that $x \in V \subseteq I(V) \subseteq U.$

Proof. Without loss of generality we may assume that
$U$ is open. Let $J \in [S]^{<\omega}$ such that $x \notin \cup J = X - U$. For
each $F \in J$ choose $F' \in S$ such that $x \in \text{int}_X(F')$ and $F' \cap F = \emptyset$.
This is possible since $S$ is normal and since $\{x\} = \cap \{s \in S | x \in S\}$ and since $S$ is binary. Then $V = \cap _{F \in J} \text{int}_X(F')$ is
as required.
2.8. Theorem. Let $X$ be a normally supercompact space. Then $\chi(X) \leq d(X) \cdot t(X)$.

Proof. Use Lemma 2.8 and the same technique as in Theorem 2.2.

References


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