SOME PROPERTIES OF WHITNEY CONTINUA

by

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1. Introduction

A continuum is a compact connected metric space. The letter $X$ will always denote a continuum with metric $d$, and $C(X)$ is the hyperspace of nonempty subcontinua of $X$ metrized by the Hausdorff metric $H$. For basic facts about hyperspaces, see [12]. If $A \in C(X)$, then $C(A) = \{Y \in C(X) | Y \subseteq A\}$ and $A = \{\{a\} | a \in A\}$. A continuous map $\mu: C(X) \to \mathbb{R}$ is called a Whitney map if it satisfies: (1) $\mu(\{x\}) = 0$ for each $x \in X$, and (2) if $A \subseteq B$ and $A \neq B$, then $\mu(a) < \mu(b)$. Whitney [16] has shown that such maps always exist. Throughout this paper, $\mu$ will stand for an arbitrary Whitney map on $C(X)$. It is known [4] that $\mu$ is monotone, i.e., $\mu^{-1}(t)$ is a subcontinuum of $C(X)$ for each $t$. The continua $\mu^{-1}(t)$ are called the Whitney continua. Notice that if $A \in C(X)$, then $C(A) \cap \mu^{-1}(t)$ is a continuum since it is a Whitney continuum in $C(A)$.

A topological property $P$ is said to be a Whitney property provided that whenever a continuum $X$ has property $P$, so does $\mu^{-1}(t)$ for each Whitney map $\mu$ for $C(X)$ and each $t$ with $0 < t < \mu(X)$. Whitney properties were investigated by several authors (see [8], [14], [15], and, for a summary of results, see [12]). Nadler [12] defines a topological

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property $P$ to be a *strong Whitney-reversible property* (resp., *Whitney-reversible property*) provided that whenever $X$ is a continuum such that $\mu^{-1}(t)$ has property $P$ for some Whitney map (resp., all Whitney maps) $\mu$ for $C(X)$, and all $t$ with $0 < t < \mu(X)$, then $X$ has property $P$. Nadler ([12], [13]) has shown that some topological properties are Whitney-reversible and he asked [12, (14.57)] if certain other properties are Whitney-reversible. In section 2 we show that hereditary decomposability, hereditary arcwise connectedness, and $C^*$-smoothness are strong Whitney-reversible properties.

In section 3 we study the relation between convexity of the Whitney continua and that of the underlying continuum.

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2. Whitney-Reversible Properties

A continuum is said to be *decomposable* provided that it is the union of two proper subcontinua. It is said to be *indecomposable* provided that it is not decomposable. A property $P$ of a continuum $X$ is said to be *hereditary* provided that each subcontinuum of $X$ has $P$. We will denote by $\sigma$ the union function $\sigma: C(C(X)) \to C(X)$ defined by $\sigma(a) = \bigcup \{A | A \in a\}$, and by $\hat{i}$ the function $\hat{i}: C(X) \to C(C(X))$ defined by $\hat{i}(A) = \hat{A}$. It is known that $\sigma$ is continuous [6], and that $\hat{i}$ is an isometry [12, (16.6)].

It is known [12, p. 413] that indecomposability is not a Whitney property. However, this result shows that indecomposability of $X$ is reflected in $\mu^{-1}(t)$.
2.1. Theorem. Let $X$ be an indecomposable continuum. Let $\mu$ be a Whitney map for $C(X)$. Then for each $t \in (0, \mu(X))$ there exists an indecomposable continuum $\beta_t \subseteq \mu^{-1}(t)$ such that $\sigma(\beta_t) = X$.

Proof. Let $t \in (0, \mu(X))$ be fixed. It follows by the continuity of the union function $\sigma$ and Brouwer's reduction theorem that $\mu^{-1}(t)$ contains a continuum $\beta_t$ which is irreducible with respect to the property that $\sigma(\beta_t) = X$. We show that $\beta_t$ is indecomposable. For, if $\beta_t$ were the union of two proper subcontinua $\beta_1$ and $\beta_2$, then $\sigma(\beta_1)$ and $\sigma(\beta_2)$ would be proper subcontinua of $X$ such that $X = \sigma(\beta_1) \cup \sigma(\beta_2)$. This contradicts the fact that $X$ is indecomposable.

It is known (see [12, p. 454]) that decomposability is not a Whitney property.

2.2. Theorem. Assume there is a sequence $\{t^n\}_{n \in \omega}$ such that $t^n \to 0$ as $n \to \omega$, and $\mu^{-1}(t^n)$ is hereditarily decomposable for each $n = 1, 2, 3, \ldots$, then $X$ is hereditarily decomposable. Hence, hereditary decomposability is a strong Whitney-reversible property.

Proof. Suppose on the contrary that $X$ contains an indecomposable continuum $Y$. It follows easily from the continuity of $\mu$, and the hypothesis of the theorem that there exists $t_0 \in \{t^n | n \in \omega\}$ such that $C(Y) \cap \mu^{-1}(t_0)$ is a non-degenerate subcontinuum of $\mu^{-1}(t_0)$. Then, by 2.1, there exists an indecomposable continuum $\beta \subseteq C(Y) \cap \mu^{-1}(t_0)$. This contradicts the fact that $\mu^{-1}(t_0)$ is hereditarily decomposable.
The result just proved answers one of the questions in [12, (14.57)].

A continuum $X$ is **unicoherent** provided that $A \cap B$ is connected whenever $A$ and $B$ are subcontinua of $X$ such that $A \cup B = X$. A **triad** is a continuum $M$ which contains a subcontinuum $N$ such that the complement of $N$ in $M$ is the union of three nonempty mutually separated sets. A continuum is **a-triodic** provided it contains no triad. A continuum $X$ is **chainable** provided that for each $\varepsilon > 0$, there exists a continuous function $f : X \to \mathbb{R}$ such that $\text{diam}(f^{-1}(r)) < \varepsilon$ for each $r \in f(X)$.

Nadler has proved the following result (see [12, (14.46), (14.49-51)].

2.3. **Theorem [Nadler].** Assume there is a sequence $\{t_n\}_{n \in \omega}$ such that $t_n \to 0$ as $n \to \infty$ and $\mu^{-1}(t_n)$ is unicoherent (or, respectively, a-triodic, an arc, a circle), then $X$ is unicoherent (or, respectively, a-triodic, an arc, a circle).

The following two results provide partial answers to the question of whether chainability is a Whitney-reversible property.

2.4. **Theorem.** Assume there is a sequence $\{t_n\}_{n \in \omega}$ such that $t_n \to 0$ as $n \to \infty$ and $\mu^{-1}(t_n)$ is an hereditarily decomposable chainable continuum for each $n = 1, 2, 3, \ldots$, then $X$ is an hereditarily decomposable chainable continuum.

**Proof.** It follows by 2.2 that $X$ is hereditarily decomposable. Since a chainable continuum is hereditarily unicoherent and a-triodic, it follows by 2.3 that $X$ is hereditarily
unicoherent and a-triodic. Bing [2, Theorem 11] has proved that an hereditarily decomposable continuum is chainable if and only if it is a-triodic and hereditarily unicoherent.

A continuum \( X \) is said to have property \([\kappa] \) provided that for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( a,b \in X \), \( d(a,b) < \delta \), and \( a \in A \subset C(X) \), then there exists \( B \in C(X) \) such that \( b \in B \), and \( H(A,B) < \varepsilon \). It is known [6] that if \( X \) has property \([\kappa] \), then the function \( F : X \times [0,\mu(X)] \rightarrow C(C(X)) \) defined by \( F(x,t) = \{ A \in \mu^{-1}(t) | x \in A \} \) is continuous.

2.5. Theorem. Let \( X \) be a continuum which has property \([\kappa] \). Assume there is a sequence \( \{ t_n \}_{n \in \omega} \) such that \( t_n \rightarrow 0 \) as \( n \rightarrow \infty \), and \( \mu^{-1}(t_n) \) is chainable for each \( n = 1,2,\ldots \), then \( X \) is chainable.

Proof. Let \( \varepsilon > 0 \) be given. By the continuity of \( \mu \), and the hypothesis of the theorem, there exists \( t_0 \in \{ t_n | n \in \omega \} \) such that \( \text{diam}(M) < \varepsilon/2 \) for each \( M \in \mu^{-1}(t_0) \). Since \( \mu^{-1}(t_0) \) is chainable, there exists a continuous map \( g: \mu^{-1}(t_0) \twoheadrightarrow [0,1] \) such that \( \text{diam}(g^{-1}(r)) < \varepsilon/2 \) for each \( r \in [0,1] \). Define \( f: X \rightarrow [0,1] \) by \( f(x) = \text{centre}(g(F(x,t_0))) \). Since \( X \) has property \([\kappa] \), \( f \) is continuous. Let \( r \in f(x) \), and let \( a,b \in f^{-1}(r) \). Then there exist \( A \in F_\mu(a,t_0) \) and \( B \in F_\mu(b,t_0) \) such that \( r = g(A) = g(B) \). Since \( g \) is an \( \varepsilon/2 \)-map, \( H(A,B) < \varepsilon/2 \). Thus, \( d(a,b) < \varepsilon \). This shows that \( f \) is an \( \varepsilon \)-map. Hence, \( X \) is chainable.

It is known (see [12, (14.48)]) that arcwise connectedness is not a Whitney-reversible property. Let us note the following:
2.6. Theorem. Assume that $\mu^{-1}(t)$ is hereditarily arcwise connected for each $t \in (0, \mu(X))$, then $X$ is an arc or a circle. Hence, hereditary arcwise connectedness is a strong Whitney-reversible property.

Proof. It is known [9, p. 212] that each arcwise connected continuum is decomposable. Thus, each $\mu^{-1}(t)$ is hereditarily decomposable for each $t \in (0, \mu(X))$. Then, by 2.2, $X$ is hereditarily decomposable. It follows by [8, (3.3)] that $X$ is $a$-triadic. Now, we show that $C(X) \setminus \{E\}$ is arcwise connected for each proper subcontinuum $E$ of $X$. Let $E$ be an arbitrary but fixed subcontinuum of $X$. We may assume that $E$ is non-degenerate. To prove that $C(X) \setminus \{E\}$ is arcwise connected, it suffices from the arc structure of $C(X)$ to show that if $A$ is a proper subcontinuum of $E$, then $A$ and $X$ can be joined by an arc in $C(X) \setminus \{E\}$. Let $t > 0$ be chosen such that $\mu(A) \leq t < \mu(E)$. Let $B \in \mu^{-1}(t)$ such that $A \subseteq B$, and let $a_1$ be an order arc from $A$ to $B$ (see [12]). Let $g \in X \setminus E$, and let $G \in \mu^{-1}(t)$ such that $g \in G$. Since $\mu^{-1}(t)$ is arcwise connected, there exists an arc $a_2$ joining $B$ and $G$ in $\mu^{-1}(t)$. Let $a_3$ be an order arc from $G$ to $X$. It follows that $a_1 \cup a_2 \cup a_3$ is an arc joining $A$ and $X$ in $C(X) \setminus \{E\}$. This shows that $C(X) \setminus \{E\}$ is arcwise connected. Since $X$ is $a$-triadic and hereditarily decomposable, it follows by [12, (11.16)] that $X$ is chainable or circle-like.

If $X$ is chainable, then since the property of being a chainable continuum is a Whitney property [7], each $\mu^{-1}(t)$ is chainable, $0 < t < \mu(X)$. Since each arcwise connected chainable continuum is an arc, each $\mu^{-1}(t)$ is an arc. Then, by 2.3, $X$ is an arc. On the other hand, if $X$ is circle-like
and not chainable (i.e., proper circle-like), then since the property of being a proper circle-like continuum is a Whitney property [7], each $\mu^{-1}(t)$ is a proper circle-like continuum, $0 < t < \mu(X)$. Thus, each $\mu^{-1}(t)$ is an hereditarily arcwise connected circle-like continuum. By [11, Theorem 6], each $\mu^{-1}(t)$ is a circle. Thus, by 2.3, $X$ is a circle.

A continuum $X$ is said to be $C^*$-smooth provided that the function $C^*: C(X) \rightarrow C(C(X))$ defined by $C^*(A) = C(A)$ is continuous [12, (15.5)].

We denote by $H^2$ the Hausdorff metric on $C(C(X))$ corresponding to $H$ as a metric on $C(X)$, and by $H^3$ the Hausdorff metric on $C(C(C(X)))$ corresponding to $H^2$ as a metric on $C(C(X))$.

2.7. Lemma. For each $\epsilon > 0$ there exists $\delta > 0$ such that if $t < \delta$, $A$ is any subcontinuum of $X$, $\mu^{-1}(t)$ is hereditarily unicoherent, and $B$ is any subcontinuum of $\mu^{-1}(t)$ such that $\sigma(B) = A$, then $H^3(C(\hat{A}), C(\hat{B})) \leq \epsilon$.

Proof. Let $\epsilon > 0$ be given. By the continuity of $\mu$ and the compactness of $C(X)$, there exists $\delta > 0$ such that if $0 < t < \delta$, and $M \in \mu^{-1}(t)$, then $\text{diam}(M) < \epsilon$. Assume that $\mu^{-1}(t)$ is hereditarily unicoherent for some $t < \delta$. Let $A \in C(X)$, and let $B \subseteq C(\mu^{-1}(t))$ such that $\sigma(B) = A$. Now,$H^3(C(\hat{A}), C(\hat{B})) = \max\{ \sup_{M \subseteq C(\hat{A})} (\inf_{N \subseteq C(\hat{B})} H^2(M, N)), \sup_{N \subseteq C(\hat{B})} (\inf_{M \subseteq C(\hat{A})} H^2(M, N)) \}.$

If $M \subseteq C(\hat{B})$, let $N = (\delta M)$. Then it is easy to see that $H^2(M, N) < \epsilon$. On the other hand, if $N \subseteq C(\hat{A})$, let $X(\sigma(N), \mu, t) = \{ G \subseteq \mu^{-1}(t) | G \cap \sigma(N) \neq \emptyset \}$. Then, by [8, (3.2)], $X(\sigma(N), \mu, t)$ is a subcontinuum of $\mu^{-1}(t)$. Let
2.8. Example. The following example shows that the assumption that \( \mu^{-1}(t) \) is hereditarily unicoherent cannot be dropped from Lemma 2.7. Let \( X \) be the unit circle, and let \( \mu \) be any Whitney map for \( C(X) \). Note that \( \mu^{-1}(t) \) is a circle for each \( t \in (0, \mu(X)) \) [7]. Let \( \varepsilon = 1/10 \). We show that for any \( t \in (0, \mu(X)) \), there exists a subcontinuum \( \beta \subseteq \mu^{-1}(t) \) such that \( \sigma(\beta) = X \), and \( H^3(C(\hat{X}), C(\beta)) > 1/10 \). Let \( t \in (0, \mu(X)) \) be arbitrary but fixed. It suffices to assume that \( \text{diam}(M) < 1/4 \) for each \( M \in \mu^{-1}(t) \). Let \( \ell > 0 \) such that \( \text{diam}(M) > \ell \) for each \( M \in \mu^{-1}(t) \). Let \( S \) be an open interval of \( X \) of length \( \ell \), and let \( X_1 = X \setminus S \). Let \( \beta = \{ M \in \mu^{-1}(t) | M \cap X_1 \neq \emptyset \} \). Then \( \beta \) is a subcontinuum of \( \mu^{-1}(t) \) such that \( \sigma(\beta) = X \). Let \( N \) be the arc of \( X \) of length \( 1 \) which contains \( S \) in its middle. It is easy to see that \( H^2(N, \gamma) > 1/10 \) for each subcontinuum \( \gamma \subseteq \beta \), and consequently \( H^3(C(\hat{X}), C(\beta)) > 1/10 \).

2.9. Theorem. Assume there is a sequence \( \{ t_n \}_{n \in \omega} \) such that \( t_n \to 0 \) as \( n \to \infty \), and \( \mu^{-1}(t_n) \) is C*-smooth for each \( n = 1, 2, \ldots \). Then, \( X \) is C*-smooth. Hence, C*-smoothness is a strong Whitney-reversible property.

Proof. Let \( \{ A_n \}_{n \in \omega} \) be a sequence in \( C(X) \) such that \( \lim A_n = A \). To prove that \( X \) is C*-smooth, it suffices to show that if \( \{ C(A_{n_j}) \}_{j \in \omega} \) is any convergent subsequence of the sequence \( \{ C(A_n) \}_{n \in \omega} \), then \( \lim_{j \to \infty} C(A_{n_j}) = C(A) \). We may assume that \( A \) is non-degenerate. Let \( A = \lim_{j \to \infty} C(A_{n_j}) \), and let \( \varepsilon > 0 \).
be arbitrary. Let $\delta > 0$ be chosen as in Lemma 2.7 with $\varepsilon$ replaced by $\varepsilon/3$. Let $t \in \{t_n \mid n \in \omega\}$ such that $t < \delta$, and such that $C(A) \cap \mu^{-1}(t)$ is a non-degenerate continuum. Then, by [5, (2.1)], $\lim_{j \to \infty} C(A_{n_j}) \cap \mu^{-1}(t) = A \cap \mu^{-1}(t)$. Since $\mu^{-1}(t)$ is $C^*$-smooth, there exists a natural number $N$ such that for each $j \geq N$,

$$H^3(C(A_{n_j}) \cap \mu^{-1}(t)), C(A \cap \mu^{-1}(t)) < \varepsilon/3.$$  

(1)

We may assume that for each $j \geq N$, $o(C(A_{n_j}) \cap \mu^{-1}(t)) = A_{n_j}$. Since each $C^*$-smooth continuum is hereditarily unicoherent [5], it follows by 2.7 that

$$H^3(C(A_{n_j}), C(A \cap \mu^{-1}(t))) < \varepsilon/3.$$  

(2)

Since the union function $o$ is continuous, $A = o(A \cap \mu^{-1}(t))$. Hence, by 2.7

$$H^3(C(\hat{A}), C(A \cap \mu^{-1}(t))) < \varepsilon/3.$$  

(3)

It follows from (1), (2), and (3) and the triangle inequality that $H^3(C(\hat{A}), C(A_{n_j})) < \varepsilon$ for each $j \geq N$. Since for each $M \in C(\hat{A})$, and each $N \in C(A_{n_j})$, $H^2(M, N) = H(o(M), o(N))$, it follows that $H^2(C(A), C(A_{n_j})) < \varepsilon$ for each $j \geq N$. Consequently, $H^2(C(A), A) < \varepsilon$. Since $\varepsilon$ is arbitrary, $A = C(A)$ and the proof is complete.

2.10. Remark. In contrast with 2.9, let us show that $C^*$-smoothness is not a Whitney property. By [12, (15.11)] a locally connected continuum is $C^*$-smooth if and only if it is a dendrite. Let $X$ be a simple triod (a continuum homeomorphic to $\{0, y \in \mathbb{R}^2 \mid 0 \leq y \leq 1\} \cup \{(x, 1) \in \mathbb{R}^2 \mid -1 \leq x \leq 1\}$). Then $X$ is $C^*$-smooth. It follows by [12, (14.9)] that $\mu^{-1}(t)$ is a
locally connected continuum for each \( t \in (0, \mu(X)) \). It is easy to see that \( \mu^{-1}(t) \) contains a 2-cell for each \( t \in (0, \mu(X)) \), and, therefore, \( \mu^{-1}(t) \) is not C*-smooth.

3. Convexity

A continuum \( X \) is said to be convex provided that for each pair of points \( x, y \in X \), there exists a point \( z \in X \setminus \{x, y\} \) such that \( d(x, z) + d(z, y) = d(x, y) \). It is known that if \( X \) is convex, then each pair of points of \( X \) can be joined by a segment in \( X \).

Let us note the following theorem for which we will show the converse is false.

3.1. Theorem. Assume there is a sequence \( \{t_n\}_{n \in \omega} \) such that \( t_n \to 0 \) as \( n \to \infty \), and \( \mu^{-1}(t_n) \) is convex (with respect to the Hausdorff metric), then \( X \) is convex (with respect to the original metric \( d \) on \( X \)).

Proof. Since \( \mu \) is an open map [4], and \( \lim_{n \to \infty} t_n = 0 \), \( \lim_{n \to \infty} \mu^{-1}(t_n) = \hat{X} \). Since each \( \mu^{-1}(t_n) \) is convex, it follows by [3, (4.8)] that \( \hat{X} \) is convex, and consequently \( X \) is convex.

3.2. Example. The following is an example of a convex arc \( X \), and a Whitney map \( \mu \) for \( C(X) \), such that \( \mu^{-1}(t) \) is not convex for any \( t \in (0, 1] \). Let \( X = [0, 3] \) with the Euclidean metric. Define a homeomorphism \( f: [0, 3] \to [0, 6] \) as follows:

\[
 f(x) = \begin{cases} 
 x, & \text{if } x \in [0, 1] \\
 x^2, & \text{if } x \in [1, 2] \\
 2x, & \text{if } x \in [2, 3].
\end{cases}
\]
Define \( \mu : C(X) \to [0, \infty) \) by \( \mu([a,b]) = f(b) - f(a) \). Then, \( \mu \) is a Whitney map for \( C(X) \). We show that \( \mu^{-1}(t) \) is not convex.

Let \( t \in (0,1] \) be fixed. Let \( A = [0,t] \), \( B = [3-t/2,3] \), and \( D = [1/\|1+t\|] \). Then \( A, B \) and \( D \in \mu^{-1}(t) \). It is known that \( \mu^{-1}(t) \) is an arc \([7]\). Note that \( A \) and \( B \) are the end points of \( \mu^{-1}(t) \). It is easy to see that \( H(A,D) = 1, H(D,B) = 3 - \sqrt{1+t} \), and \( H(A,B) = 3 - t/2 \). Thus, \( H(A,B) \neq H(A,D) + H(D,B) \). This shows that \( \mu^{-1}(t) \) is not convex.

3.3. Remark. It is known \([1]\) that a convex continuum is locally connected, and that local connectedness is a Whitney property \([12, (14.9)]\). Bing \([1]\) and Moise \([10]\) have shown independently that every locally connected continuum admits a convex metric. In view of these facts, we see that if \( X \) is a convex continuum, \( \mu^{-1}(t) \) admits a convex metric. However, as 3.2 shows, it may happen that \( \mu^{-1}(t) \) is not convex with respect to the Hausdorff metric.

References


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