ON THE RECENT DEVELOPMENT OF
THE NIELSEN FIXED POINT THEOREMS
OF FIBER-PRESERVING MAPS

by

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1. Introduction

Let \( f: X \rightarrow X \) be a continuous map on a compact connected ANR \( X \) into itself. The Nielsen fixed point theorem says that every map \( g: X \rightarrow X \) homotopic to \( f \) has at least \( N(f) \), the Nielsen number of \( f \), fixed points. Thus the Nielsen fixed point theorem is more powerful than the Lefschetz fixed point theorem which only ensures the existence of a single fixed point if the Lefschetz number of \( f \), \( L(f) \neq 0 \). If \( L(f) = 0 \) then no conclusion can be drawn as to whether or not there exists a fixed point. In fact, recently McCord [9] has constructed a homeomorphism \( h \) on a manifold \( M^n \) onto itself such that \( L(h) = 0 \) and \( N(h) \geq 2 \) in all dimensions \( n \).

The purpose of this paper is to introduce recent developments of the product theorems for the Nielsen numbers of a fiber-preserving map. For the convenience of readers we introduce the Nielsen fixed point theorems from [1] and [4] in section 2. In the third section we study the Jiang's contribution [8] to estimate the Nielsen numbers of a continuous map. In the fourth section we cover some recent results dealing with the product theorems of the Nielsen number of a fiber-preserving map. In particular, we show that the recent product theorem of Giessmann [6] reduces to that of Pak [11].
There are a couple of excellent articles on the Nielsen fixed point theorems by Brown [1] and Fadell [4]. Therefore, we suggest to the readers to these publications for more materials and details with regard to the sections 2 and 3.

2. The Nielsen Fixed Point Theorems

Let \( f: X \rightarrow X \) be a continuous map on a compact connected ANR \( X \) into itself. Let \( \phi(f) = \{ x \in X | f(x) = x \} \) be the set of all fixed points of \( f \). Any two elements \( x, y \in \phi(f) \) are said to be \( f \)-equivalent if there is a path \( C: I \rightarrow X \) such that \( C(0) = x, C(1) = y \), and \( C \simeq f(C) \) (homotopic). This relation is an equivalence relation in \( \phi(f) \) and divides \( \phi(f) \) into finite number of equivalence classes \( F \). If the fixed point index \( i(F) \neq 0 \) then \( F \) is called essential Nielsen fixed point class and if \( i(F) = 0 \) then \( F \) is called inessential. It is known that if \( i(F) = 0 \) then we could remove the fixed points in \( F \) by a map \( g \) homotopic to \( f \) in many cases.

**Definition.** The Nielsen number \( N(f) \) of a map \( f \) is defined to be the number of essential fixed point classes of \( f \).

**Theorem [Nielsen].** Let \( f: X \rightarrow X \) be a continuous map from a compact connected ANR \( X \) into itself. If \( N(f) \neq 0 \) then every map \( g: X \rightarrow X \) homotopic to \( f \) has at least \( N(f) \) fixed points.

In many cases, stronger conclusions can be drawn. For example, if \( X \) is a manifold of dimension \( \geq 3 \) then there is a map \( g \) homotopic to \( f \) which has exactly \( N(f) \) fixed points.
3. On the Jiang Spaces

In an effort to compute the Nielsen number of a given map \( f : X \to X \), Jiang [8] introduced an interesting subgroup of the fundamental group of \( X \). Let \( M(X) \) be the space of all continuous maps from a compact connected ANR \( X \) into itself with compact open topology. Let \( \alpha : M(X) \to X \) be a map defined by \( \alpha(f) = f(x_0) \), i.e., the evaluation map at \( x_0 \in X \). Then \( \alpha \) induces \( \alpha_* : \pi_1(M(X), f) \to \pi_1(X, f(x_0)) \). The Jiang subgroup \( T(X, f, x_0) \) of \( f \) is defined to be the image \( \alpha_* (\pi_1(X, f)) \) in \( \pi_1(X, f(x_0)) \). If we denote \( T(X) \) for \( T(X, \text{id}, x_0) \), then \( T(X) \subseteq T(X, f, x_0) \subseteq \pi_1(X, f(x_0)) \) for all \( f \in M(X) \). It is known that \( T(X) \) lies in the center of \( \pi_1(X, x_0) \) and if \( T(X) = \pi_1(X, x_0) \) then \( \pi_1(X, x_0) \) abelian, and \( X \) said to satisfy the Jiang condition (\( J \)-condition). It is well known lens spaces and \( H \)-spaces satisfy the \( J \)-condition.

**Theorem [Jiang].** If \( X \) satisfies the \( J \)-condition then each Nielsen fixed point class \( F \) of \( f \) has the same fixed point index \( i(F) \) and if we denote this number by \( i(f) \) then \( L(f) = i(f) \cdot N(f) \).

**Definition.** Let \( f_* : \pi_1(X, x_0) \to \pi_1(X, x_0) \) be a homomorphism. Two elements \( \alpha \) and \( \beta \) are said to be \( f \)-equivalent if there exists \( \gamma \in \pi_1(X, x_0) \) such that \( \alpha = \gamma \beta f_* (\gamma^{-1}) \). The Riedeweister number \( R(f) \) of \( f \) is defined to be the cardinality of the set of equivalence classes in \( \pi_1(X, x_0) \).

We apply the \( f \)-equivalence relation to \( T(f) \) and denote the cardinality of equivalence classes by \( J(f) \).

**Theorem [Brooks, Brown, Jiang].** Assume \( L(f) \neq 0 \). Then
\( J(f) \leq N(f) \leq R(f). \)

**Theorem [Jiang].** Assume \( L(f) \neq 0 \). If \( T(X) = \pi_1(X) \), then \( N(f) = J(f) = R(f) \).

### 4. On the Fiber-Preserving Maps

To compute the Nielsen number of a given map \( f \), Brown [2] initially studied fiber-preserving maps. Let \( J = \{E,p,B\} \) be an orientable Hurewicz fibering with a regular lifting function \( \lambda \). If \( f: E \to E \) is a fiber-preserving map then \( f \) induces \( f': B \to B \) and \( f_b: P^{-1}(b) \to P^{-1}(b) \) defined by \( f_b(e) = \lambda(e,w)(1) \) where \( w \) is a path from \( f(b) \) to \( b \in B \). It is well known that \( N(f_b) \) is independent of the choice of \( w \) and \( b \in B \). The product theorem for the Nielsen number of a fiber-preserving map says \( N(f) = N(f') \cdot N(f_b) \), \( b \in B \), under the suitable conditions. Thus for a class of fiber-preserving maps we may compute the Nielsen number through the product theorem. Note that \( L(f) = L(f') \cdot L(f_b) \) holds for all \( b \in B \).

**Theorem [Brown-Fadell].** Let \( J = \{E,p,B\} \) be an orientable Hurewicz fibering with fiber \( Y \) where \( E, B \) and \( Y \) are connected finite polyhedra and let \( f: E \to E \) be a fiber-preserving map. If one of the following conditions is satisfied

- (a) \( \pi_1(B) = \pi_2(B) = 0 \)
- (b) \( \pi_1(Y) = 0 \)
- (c) \( J \) is trivial and either \( \pi_1(B) = 0 \) or \( f = f' \times f_b \)

then \( N(f) = N(f') \cdot N(f_b) \) for all \( f_b, b \in B \).

Recently Fadell [5] came up with a unifying theorem for a number of existing theorems.
Theorem [Fadell]. Let $J = \{E, p, B\}$ be an orientable Hurewicz fibering with $E$, $B$ and $Y$ compact connected ANR's and let $f: J \rightarrow J$ be a fiber-preserving map. Suppose $f$ and $J$ satisfy the following conditions:

a) The sequence
$$0 \rightarrow \pi_1(Y) \xrightarrow{i^\#} \pi_1(E) \xrightarrow{p^\#} \pi_1(B) \rightarrow 0$$
is exact,

b) $p^\#$ admits a right inverse $\sigma: \pi_1(B) \rightarrow \pi_1(E)$ such that
if $H = \text{im } \sigma$, then $H$ is normal in $\pi_1(E)$ and $f^\#(H) \subseteq H$.

Then $N(f) = N(f') \cdot N(f_b)$, $b \in B$.

If spaces involved in $J$ satisfy the Jiang condition then we have a complete solution to the product theorem.

Without loss of generality we may assume $L(f) \neq 0$. Consider the following part of fiber homotopy exact sequence:

$$
\begin{array}{c}
\pi_1(P^{-1}(b)) \\
\downarrow 1-f_b^\# \\
\pi_1(P^{-1}(b))
\end{array} \xrightarrow{i^\#} \pi_1(E) \xrightarrow{1-f^\#} \pi_1(E)
$$

This diagram induces a homomorphism
$$i^\#: \text{coker}(1-f_b^\#) \rightarrow \text{coker}(1-f^\#).$$

Definition. Define $P(f)$ to be the order of $\ker i^\#$.

Theorem [Pak]. Let $J = \{E, p, B\}$ be an orientable Hurewicz fibering. We assume $E$, $B$ and $Y$ satisfy the $J$-condition. If $f: J \rightarrow J$ is a fiber-preserving map then $N(f) \cdot P(f) = N(f') \cdot N(f_b)$, $b \in B$.

The fiber homotopy exact sequence induces the following diagram:
where $K$ is kernel and $C$ is cokernel of corresponding maps, and $d\# : \pi_2(B) \to \pi_1(P^{-1}(b))$ the connecting homomorphism. The sequence $C$ is right exact, i.e., exact at $C_1$ and $C_2$ and $K$ is left exact, i.e., exact at $K_4$ and $K_3$. Also the homology groups satisfy $H(C) = H(K)$ and $H(C) = H(K)$. Let us denote the order of a group $G$ by $\text{O}(G)$.

In an effort to generalize Pak's theorem Giessmann [6] proved the following theorem.

**Theorem [Giessmann].** Let $J = \{E, p, B\}$ be an orientable Hurewicz fibering. Let $f: J \to J$ be a fiber-preserving map. If $E$ satisfies $J$-condition and if $\text{Im}(1-f_b\#)$ and $\text{Im}(1-f_b')$ lie in the corresponding Jiang subgroups $T(f_b)$ and $T(f')$ respectively then

$$N(f) \cdot Q(f) = N(f') \cdot N(f_b) \cdot R(f),$$

where $Q(f) = 0(H(K_1)) \cdot 0(C_4)$ and $R(f) = 0(H(K_4)).$

**Lemma 1.** $0(C_2) \cdot P(f) = 0(C_1) \cdot 0(C_3)$.

**Proof.** The above diagram induced by the fiber homotopy exact sequence induces the following short exact sequences:

$$0 \to H(C_4) \to C_4 \to \text{Im} \alpha \to 0,$$

$$0 \to \ker \gamma \to C_1 \to \text{Im} \beta \to 0,$$

$$0 \to \ker \iota \to C_2 \to C_3 \to 0,$$

and

$$0 \to \ker \gamma \to C_1 \to \text{Im} \beta \to 0,$$

$$0 \to \ker \iota \to C_2 \to C_3 \to 0.$$
we have

$$0 \rightarrow \ker i^\# \rightarrow C_3 \rightarrow \ker \gamma \rightarrow C_2 \rightarrow C_1 \rightarrow 0.$$  

From the above exact sequences we can read off

$$0(C_4) = 0(H(C_4)) \cdot 0(\operatorname{Im} \alpha),$$
$$0(C_3) = 0(\ker i^\#) \cdot 0(\operatorname{Im} i^\#),$$
$$0(C_2) = 0(C_1) \cdot 0(\ker \gamma)$$
and

we have

$$0(C_3) \cong 0(\operatorname{Im} i^\#) = 0(\ker \gamma) = 0(C_2)$$

$$0(\ker i^\#) \quad 0(C_1).$$

Therefore we get $0(C_2) \cdot 0(\ker i^\#) = 0(C_1) \cdot 0(C_3)$. Since $P(f) = 0(\ker i^\#)$ our lemma is proved.

**Theorem 2.** Let $J = (E, p, B)$ be an orientable Hurewicz fibering and let $f: J \rightarrow J$ be a fiber-preserving map such that $L(f) \neq 0$. If $E$ satisfies the $J$-condition and if $\operatorname{Im}(1-f_b i^\#)$ and $\operatorname{Im}(1-f'_i)$ lie in the corresponding Jiang subgroups $T(f_b)$ and $T(f')$, then $N(f) \cdot P(f) = N(f') \cdot N(f_b), \quad b \in B.$

**Proof.** Giessmann's theorem says $N(f) \cdot Q(f) = N(f') \cdot N(f_b) \cdot R(f)$. Since we know $0(H(K_2)) = 0(H(C_4))$ and $0(H(K_1)) = 0(H(C_3))$ we have $N(f) \cdot 0(H(C_3)) \cdot 0(C_4) = N(f_b) \cdot N(f') \cdot 0(H(C_4)).$ From the proof of lemma 1, we have

$$N(f) \cdot 0(H(C_3)) \cdot 0(H(C_4)) \cdot 0(\operatorname{Im} \alpha) = N(f_b) \cdot N(f') \cdot 0(H(C_4)),$$
and this reduced to $N(f) \cdot 0(\frac{\ker i^\#}{\operatorname{Im} \alpha}) \cdot 0(\operatorname{Im} \alpha) = N(f_b) \cdot N(f').$

Thus we conclude $N(f) \cdot P(f) = N(f') \cdot N(f_b), \quad b \in B.$

Finally we would like to pose a couple of problems.

**Problem 1.** If $E = B \times Y$, then Gottlieb [7] has shown that $E$ satisfies the $J$-condition iff $B$ and $Y$ satisfy the
J-condition. Derivation of theorem 2 from that of Giessmann strongly suggests the following problem:

Let \( \mathcal{J} = \{E,p,B,Y\} \) be an orientable Hurewicz fibering. Then \( E \) satisfies the J-condition iff \( B \) and \( Y \) satisfy the J-condition.

**Problem 2.** It is known that simply connected spaces, lens spaces, and H-spaces satisfy the J-condition. It will be an interesting problem to try to enlarge the class of Jiang spaces.

*Added in Proof.* Recently we have shown "only if" part of problem 1. That is if \( E \) satisfies J-condition then \( B \) and \( Y \) satisfy the J-condition. This result gives:

**Theorem 3.** Let \( \mathcal{J} = \{E,p,B\} \) be an orientable Hurewicz fibering, where \( E \) satisfies the J-condition. If \( f: \mathcal{J} \to \mathcal{J} \) is a fiber-preserving map such that \( L(f) \neq 0 \), then \( N(f) \cdot P(f) = N(f') \cdot N(f_b) \), \( b \in B \).

A proof of this theorem also follows easily from the following theorem of Fadell [5]. We leave the proof to the reader.

*Theorem [Fadell].* Let \( \mathcal{J} = \{E,p,B\} \) be a Hurewicz fibering with \( E \) and \( B \) compact, metric ANR's and \( f: E \to E \) a fiber-preserving map. Then there is a locally trivial fibering \( \mathcal{J}' = \{E',p',B'\} \) with fiber \( F' \) and a fiber-preserving map \( g: E' \to E' \) with the following properties:

1) \( B' \) and \( F' \) are compact polyhedra,
2) \( g': B' \to B' \) has precisely \( N(g') \) fixed points, each
in a maximal simplex,

3) For each \( b \in \Phi(g') \), \( g_b: F' \to F' \) has precisely \( N(g_b) \) fixed points, each in a maximal simplex of \( F' \),

4) \( N(f) = N(g), N(f') = N(g'), \) and \( N(f_b) = N(g_b) \).

This theorem implies that without loss of generality we may assume \( f \) has all the properties of \( g \). Let \( \{F_1, \ldots, F_k\} = \Phi(f) \) be the Nielsen fixed point classes of \( f \). If \( \ell_{i,1} \) and \( \ell_{i,2} \in F_i \) then \( P(\ell_{i,1}) \sim P(\ell_{i,2}) \) in \( B \) and from the fact that \( f' \) has exactly \( N(f') \) fixed points we deduce that all points in \( F_i \) lie in the same fiber say \( P^{-1}(b) \). Let \( F_i \cap P^{-1}(b) = \{\ell_{i,1}, \ldots, \ell_{i,k_i}\} \). Define a map \( \gamma: \Phi(f_b) \to \Phi(f) \) for each \( b \in B \) by \( \gamma(\ell_{i,j}) = F_i \). This is a well-defined map. Let \( P(f)_{F_i} = \#(\gamma^{-1}(F_i)) = k_i \). Then the following theorem follows easily.

**Theorem 4.** Let \( J = (E,p,B) \) be an orientable Hurewicz fibering and \( f: E \to E \) a fiber-preserving map. If \( P(f) = \#(\gamma^{-1}(F_i)) \) is a constant for each \( i \), then \( N(f) \cdot P(f) = N(f') \cdot N(f_b), b \in B \).

Finally I would like to state a recent theorem from [15].

**Theorem 5.** Let \( J = (E,p,B) \) be an orientable Hurewicz fibering and \( f: E \to E \) a fiber-preserving map such that \( L(f) \neq 0 \). If the fundamental groups of the spaces involved in \( J \) are abelian then \( N(f) \cdot p(f) = N(f') \cdot N(f_b), b \in B \).

Note that this theorem improves theorem 3.
References

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