NO INFINITE-DIMENSIONAL LOCALLY COMPACT ANR IS A TOPOLOGICAL GROUP

by

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The combined results of Gleason [2] and Montgomery and Zippin [4] characterize Lie groups as the locally connected, locally compact metrizable topological groups of finite dimension. The assumption of finite-dimensionality is essential here, as the example of infinite product of circles shows. In this note we remark that a theorem of Iwasawa and results from Q-manifold theory allow this assumption to be replaced by the one that the space underlying the group be an Absolute Neighbourhood Retract:

**Theorem.** Let $X$ be a locally compact, metrizable topological group. If $X \in \text{ANR}$ then $\dim X < \infty$ and hence $X$ is a Lie group.

In the proof we need the following (we write $I = [0,1]$):

**Lemma.** Let $Y \in \text{ANR}$. If there is an open cover $\mathcal{U}$ of $Y$ consisting of sets having the disjoint n-cube property, then $Y$ has the disjoint n-cube property, i.e. any map $I^n \times \{1,2\} \to Y$ is the uniform limit of maps sending $I^n \times 1$ and $I^n \times 2$ to disjoint sets.

**Proof.** (By induction on $n$). Fix $f: I^n \times \{1,2\} \to Y$, choose $\varepsilon > 0$ so that each subset of $\text{im}(f)$ of diameter $< \varepsilon$ is contained in an element of $\mathcal{U}$ and let $J$ be a triangulation of $I^n$ such that $\text{diam } f(\sigma) < \varepsilon/3$ for $\sigma \in J \times \{1,2\}$. By
inductive assumption and properties of ANR's we may assume that \( f(A \times 1) \cap f(A \times 2) = \emptyset \), where \( A \) is the \((n-1)\)-skeleton of \( J \). Let \( \{\sigma_1, \ldots, \sigma_k\} \) be all the \( n \)-simplices in \( J \). We construct maps \( f_1, \ldots, f_k : I^n \times \{1,2\} \rightarrow Y \) such that, for \( i \leq k \), the following holds

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\begin{align*}
(1)_i & \quad f_i((\sigma_1 \cup \ldots \cup \sigma_i) \times 1) \cap f_i(I^n \times 2) = \emptyset, \\
(2)_i & \quad f_i(x) = f(x) \text{ for } x \in A \times \{1,2\}, \\
(3)_i & \quad \text{dist}(f_i, f_{i-1}) < \delta/k,
\end{align*}
\]

where \( f_0 = f \). Then \( f_k \) sends \( I^n \times 1 \) and \( I^n \times 2 \) to disjoint sets and approximates \( f \) within a given \( \delta > 0 \).

The construction of \( f_i \) (We assume \( \delta < \varepsilon/3 \)): Consider the set \( J = \{j : f_{i-1}(\sigma \times 2) \cap f_{i-1}(\sigma_i \times 1) \neq \emptyset \} \). By (3) we have \( \text{dist}(f, f_{i-1}) < \varepsilon/3 \) whence \( \text{diam} f(\sigma) < \varepsilon/3 \) for \( \sigma \in J \times \{1,2\} \) and, with \( F = \bigcup_{j \in J} f_{i-1}(\sigma_j \times 2) \cup f_{i-1}(\sigma_i \times 1) \) we have \( \text{diam} F < \varepsilon \). Thus \( F \) is contained in a member of \( U \) and we may alter \( f_{i-1} \) on \( \bigcup_{j \in J} \sigma_j \times 2 \cup \sigma_i \times 1 \) modulo \( A \times \{1,2\} \) by so small an amount that the resulting map satisfies \((1)_i \) and \((3)_i \).

**Proof of the Theorem.** Assume that \( X \in \text{ANR} \) and \( \dim X = \infty \). Given an integer \( n \) it follows from a theorem of Iwasawa that each point \( x \in X \) has a neighbourhood homeomorphic to \( V_x \times \mathbb{R}^{2n+1} \), for some space \( V_x \). (See [3] or [5], p. 184). Since \( V_x \times \mathbb{R}^{2n+1} \) has the disjoint cube property (by general position applied to \( \mathbb{R}^{2n+1} \)) it follows from the Lemma that \( X \) also has this property. A locally compact ANR having the disjoint \( n \)-cube property for each \( n \) is a manifold modeled on the Hilbert cube \( Q \) [6], and hence \( X \) is a \( Q \)-manifold. This, however, is impossible since no \( Q \)-manifold carries a
topological group structure (see [1]). Thus either $X \not\in \text{ANR}$ or $\dim X < \infty$.

The above result shows in particular that, among locally compact metrizable topological groups, the property of being an ANR forces the underlying space to have a manifold structure. It is unknown if the same is true for complete metrizable groups $X$, where by a manifold we now mean a space locally homeomorphic to a Hilbert space of infinite dimension. (Even the case of linear metric spaces is not settled and in general it is known only that $X \times \ell^2$ is a manifold, cf[7]). It is also unknown if the assumption "$X \in \text{ANR}$" in the theorem can be replaced by "$X$ is locally contractible".

References

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