A COMPLEMENT THEOREM FOR CONTINUA IN A MANIFOLD

by

I. IVANŠIC AND R. B. SHER
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1. Introduction

Since the appearance of the well known complement theorem of Chapman [C] for compacta in the Hilbert cube, a number of analogous results have been obtained for compacta in euclidean space $E^n$. In [I-S-V], connectivity conditions are used to obtain a complement theorem for ILC embedded continua of cofundamental dimension 3 in $E^n$, $n > 5$; this theorem subsumes most of the previously known results in this area (many of which are listed in the bibliography of [I-S-V]).

Here we use some of the techniques developed in [I-S-V] to establish a complement theorem (Theorem 3) in manifolds other than $E^n$. Our main tool is Theorem 1, in which we obtain nice defining sequences for certain continua in piecewise linear manifolds; this should be of further use in studying problems involving embedded continua. In Section 4 we use these results to establish a piecewise linear embedding-up-to-homotopy result (Theorem 4) and obtain as a consequence a result on the existence of deleted product neighborhoods.

We assume that the reader is familiar with the basic notions of shape theory, as found for example in [B] or [D-S], and some of the basic techniques of piecewise linear topology as found in [H] or [Z].

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1 This research was carried out while the first named author was visiting the Univ. of N.C. at Greensboro.
If $X$ is a compactum in the piecewise linear manifold $M$, then $X$ is said to satisfy the \textit{inessential loops condition}, ILC, if for each neighborhood $U$ of $X$ in $M$ there exists a neighborhood $V$ of $X$ in $U$ such that each loop in $V - X$ which is nullhomotopic in $V$ is also nullhomotopic in $U - X$. The \textit{fundamental dimension} of the compactum $X$ is $\min \{ \dim Y : \text{Sh}(X) = \text{Sh}(Y), Y \text{ a compactum} \}$. In $[V_2]$, ILC was studied as it relates to the problem of finding a small polyhedral neighborhood of $X$ having spine whose dimension does not exceed the fundamental dimension of $X$.

All continua considered in this paper will be pointed 1-movable. It follows from Theorem 7.1.3 of $[D-S]$ that shape morphisms between such continua may be regarded as \textit{pointed} morphisms. We use this fact throughout, assuming that \textit{all} shape morphisms are pointed; however, we shall suppress base points from our notation.

Finally, let us recall that a map $f : X \to Y$ between ANR's is $r$-\textit{connected} if $f^* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism when $0 \leq i \leq r - 1$ and an epimorphism when $i = r$. With this in mind, we say that the shape morphism $f : X \to Y$ between pointed 1-movable continua is \textit{shape $r$-connected} if $f^* : \text{pro-}\pi_i(X) \to \text{pro-}\pi_i(Y)$ is an isomorphism of pro-groups for $0 \leq i \leq r - 1$ and an epimorphism for $i = r$. We also recall that a pro-group $G = \{ G_\alpha, g_{\alpha\beta}, A \}$ is \textit{stable} if $G$ is isomorphic in the category pro-groups to a group, and that $G$ satisfies the \textit{Mittag-Leffler condition} if for each $\alpha \in A$ there exists $\beta \geq \alpha$ such that for all $\gamma \geq \beta$, $g_{\alpha\gamma}(G_\gamma) = g_{\alpha\beta}(G_\beta)$.
2. Defining Sequences for ILC Embedded Continua

If $X$ is a compactum lying in the interior of the piecewise linear $n$-manifold $M$, a *defining sequence* for $X$ is a sequence $\{U_i\}_{i=1}^\infty$ of compact piecewise linear $n$-manifolds in $M$ such that $X = \bigcap_{i=1}^\infty U_i$ and, if $j = 1, 2, \ldots$, $U_{j+1} \subseteq \text{int } U_j$.

In Theorem 2 of [I-S-V] it was shown that under certain conditions an $r$-shape connected continuum lying in the interior of a piecewise linear manifold has a defining sequence whose members are $r$-connected. The following provides a generalization. We call a defining sequence $\{U_i\}_{i=1}^\infty$ *$r$-connected* if for $j = 1, 2, \ldots$, the inclusion of $U_{j+1}$ into $U_j$ is an $r$-connected mapping.

**Theorem 1.** Suppose $X$ is a continuum of fundamental dimension at most $k$ lying in the interior of the piecewise linear $n$-manifold $M$ and satisfying ILC, where $n \geq 5$ and $k \leq n - 3$. Suppose $\text{pro-} \pi_i(X)$ is stable for $0 \leq i \leq r - 1$ and satisfies the Mittag-Leffler condition for $i = r < n - 3$. Then there exists an $r$-connected defining sequence $\{U_i\}_{i=1}^\infty$ for $X$ such that if $j = 1, 2, \ldots$, then $U_j$ has a spine of dimension at most $k' = \max(k, r+1)$.

**Proof.** Fix $s$, $0 \leq s < r$, and inductively assume that there exists an $s$-connected defining sequence $\{V_i\}_{i=0}^\infty$ for $X$ such that if $j = 0, 1, 2, \ldots$, then $V_j$ has a spine $K_j \subseteq \text{int } V_j$ of dimension at most $k'$. The induction begins, when $s = 0$, by Theorem 4.1 of [V2]. Since $\text{pro-} \pi_s(X)$ is stable, an easy argument using Theorems 6 and 7 of [M] shows that the inclusion of $V_{j+1}$ into $V_j$ induces an isomorphism of $\pi_s(V_{j+1})$ onto $\pi_s(V_j)$. Let $r_j: V_j \to K_j$ denote the retraction induced by a
collapse of $V_j$ onto $K_j$, and let $s_j = r_j|_{K_{j+1}}$. Since 
pro-$\pi_{s+1}(X)$ satisfies the Mittag-Leffler condition we may 
assume, by taking a subsequence if necessary, that if 
j = 1, 2, \cdots, then $(s_j)^\#$ carries $(s_{j+1})^\#(\pi_{s+1}(K_{j+2}))$ onto 
$(s_j)^\#(\pi_{s+1}(K_{j+1}))$. By the proof of Theorem 4 of [F] (cf. 
Theorem 3.1 of [K] and Theorem 5 of [H-I]) there exist com-
pact polyhedra $L_0, L_1, L_2, \cdots$ and mappings $g_j: L_{j+1} \to L_j$ such 
that if $j = 0, 1, 2, \cdots$, then

1. $K_j \subset L_j$,
2. $g_j(L_{j+1}) \subset K_j$,
3. $g_{j+1} = i_j s_j$, where $i_m: K_m \to L_m$ denotes the 
inclusion, $m = 0, 1, 2, \cdots$,
4. $(i_j s_j)^\#: \pi_{s+1}(K_{j+1}) \to \pi_{s+1}(L_j)$ is an epimorphism, 
and
5. $L_j$ is obtained from $K_j$ by attaching finitely many 
$(s+2)$-cells.

Now fix $j \ge 1$, and define $h_{j-1}: L_j \to V_{j-1}$ by $h_{j-1}(x) = 
g_{j-1}(x)$ for all $x \in L_j$. Then $h_{j-1}(x) = s_{j-1}(x)$ for all 
x $\in K_j$. Noting that $s_{j-1} = \text{id}_{K_j}$ in $V_{j-1}$, it follows from 
the Borsuk Homotopy Extension Theorem, (5.13) on pg. 22 of 
[B], that there exists a map $k_{j-1}: L_j \to \text{int } V_{j-1}$ such that 
k_{j-1} = h_{j-1}$ and $k_{j-1}(x) = x$ for all $x \in K_j$. We may further 
assume that $k_{j-1}$ is piecewise linear and in general position. 
Note that $k_{j-1}$ is $s$-connected and has singular set of dimension at most $s - 1 < n - 5$; Theorem 4.3 of [St] thus applies, 
and yields an at most $k'$-dimensional polyhedron $P_j \subset \text{int } V_{j-1}$ 
such that $k_{j-1}(L_j) \subset P_j$ and the map $f_j: L_j \to P_j$ defined by 
f_j(x) = k_{j-1}(x) for all $x \in L_j$ is a simple homotopy
equivalence. Since $K_j \subseteq P_j$, there exists a regular neighborhood $W_j$ of $P_j$ such that $\text{int } V_{j-1} \supseteq W_j \supseteq \text{int } W_j \supseteq V_j$.

It is claimed that the defining sequence $\{W_i\}_{i=1}^\infty$ is $(s+1)$-connected, thereby allowing us to continue our induction. If $j = 1, 2, \ldots$, let $\beta$ denote the inclusion of $W_{j+1}$ into $W_j$. By our construction $\beta$ induces an isomorphism of $\pi_i(W_{j+1})$ onto $\pi_i(W_j)$ for $1 \leq i \leq s$, and so it remains to be shown that $\beta$ induces an epimorphism of $\pi_{s+1}(W_{j+1})$ onto $\pi_{s+1}(W_j)$. To verify the latter, consider the following diagram.

By (4), the fact that $f_j$ is a homotopy equivalence, and the fact that $W_j$ is a regular neighborhood of $P_j$, it follows that $(\alpha, \beta, \gamma, \delta)\#: \pi_{s+1}(K_j+1) \rightarrow \pi_{s+1}(W_j)$ is an epimorphism. This, along with the homotopy commutativity of the outermost rectangle of the diagram, shows that $\beta\#: \pi_{s+1}(W_{j+1}) \rightarrow \pi_{s+1}(W_j)$ is an epimorphism.

In Theorem 1 we hypothesize that $\text{pro-}\pi_i(X)$ is stable for $0 \leq i \leq r - 1$ and satisfies the Mittag-Leffler condition for $i = r$. This is equivalent to assuming that $\text{pro-}\pi_i(X)$ is stable for $0 \leq i \leq r - 1$ and that $X$ is pointed $r$-movable. These conditions hold when $X$ has shape finite $r$-skeleton, and the converse holds provided $r \geq 2$ (cf. Theorem 5 of [H-I])
the converse does not hold when $r = 1$, as seen by letting $X$ be the "Hawaiian earring" (pg. 100 in [D-S]).

We now make two brief and easy observations which shall be required in the next section.

**Observation 1.** If $\{U_i\}_{i=1}^{\infty}$ is an $r$-connected defining sequence for the continuum $X$ and $i = 1, 2, \ldots$, then the inclusion of $X$ into $U_i$ is shape $r$-connected. Hence, under the hypothesis of Theorem 1, $X$ has arbitrarily small neighborhoods $U$ in $M$ such that the inclusion of $X$ into $U$ is shape $r$-connected.

**Observation 2.** Suppose $U$ and $V$ are ANR's, $X$ is a continuum, $X \subseteq V \subseteq U$, and the inclusion of $X$ into each of $V$ and $U$ is shape $r$-connected. Then the inclusion of $V$ into $U$ is $r$-connected.

Finally, we note in the following that Theorem 1 may be improved in the case $k = 1 = r$ by obtaining $k' = 1$.

**Theorem 2.** Suppose $X$ is a pointed 1-movable continuum of fundamental dimension $k \leq 1$ lying in the interior of the piecewise linear $n$-manifold $M$ and satisfying ILC, where $n \geq 5$. Then there exists a 1-connected defining sequence $\{U_i\}_{i=1}^{\infty}$ for $X$ such that if $j = 1, 2, \ldots$, then $U_j$ has a spine of dimension $k$.

**Proof.** If $k = 0$, then $X$ is cellular in $M$. If $k = 1$, let $U$ be a compact piecewise linear manifold neighborhood of $X$ having 1-dimensional spine. By uniqueness of regular neighborhoods, $U$ is an $n$-cell with finitely many 1-handles.
Then $U$ embeds in $E^n$, so we may simply assume $X \subset E^n$. By Theorem 7.3.3 of [D-S], there exists a bouquet of circles $Y \subset E^2 \subset E^n$ such that $\text{Sh}X = \text{Sh}Y$. It is easy to verify that $Y$ has a 1-connected defining sequence in $E^n$ each member of which has 1-dimensional spine. The proof of the complement theorem (e.g. Theorem 1 of [V_1]) shows, as noted in Section 6 of [I-S-V], that $X$ also has such a defining sequence.

3. A Complement Theorem for Continua in a Piecewise Linear Manifold

The main result of this section is the complement theorem, Theorem 3 below. It generalizes one part of Theorem A of [I-S-V], and its proof is essentially the same as the proof of that theorem, only using the following result in place of Lemma 1 of [I-S-V]. This result and those that follow use the notion of relative shape, treated in [C].

**Lemma 1.** Let $X_1$ and $X_2$ be continua in the interior of the piecewise linear $n$-manifold $M$ such that for $j = 1$ or 2, $X_j$ has fundamental dimension at most $k$, $X_j$ satisfies ILC, and $\text{pro-}\pi_i(X_j)$ is stable for $0 \leq i \leq r - 1$ and satisfies the Mittag-Leffler condition for $i = r < n - 3$, where $n \geq \max (2k + 2 - r, k + 3, 5)$. Let $F = \{f_i, X_1, X_2, G\}$ and $F' = \{f_i', X_2, X_1, H\}$ be relative fundamental sequences in $M$ such that $f'f = \text{id}_{X_1}$ and $ff' = \text{id}_{X_2}$. Let $U_0$ be a compact piecewise linear manifold neighborhood of $X_1$ for which the inclusion of $X_1$ into $U_0$ is shape $r$-connected, and let $h: M \to M$ be a piecewise linear homeomorphism homotopic to the identity such that $X_2 = \text{int} h(U_0)$ and such that $h^{-1}|W_0 = f_i'|W_0$ in $U_0$ for some neighborhood $W_0$ of $X_2$ and for almost all $i$. Then for every
neighborhood $V_0$ of $X_2$, there exist a compact piecewise linear manifold neighborhood $V$ of $X_2$ lying in $V_0 \cap h(U_0)$ for which the inclusion of $X_2$ into $V$ is shape $r$-connected, a piecewise linear homeomorphism $q: M \rightarrow M$ homotopic to the identity, and a neighborhood $U_1$ of $X_1$ such that

1. $q|_M - U_0 = h|M - U_0$,
2. $q(X_1) \subseteq V$, and
3. $q|_{U_1} = f_i|_{U_1}$ in $V$ for almost all $i$.

Proof. As in the proof of Lemma 3 of [I-S-V], we note that the proof of Lemma 1 of [I-S-V] will apply provided we can find an arbitrarily small compact piecewise linear manifold neighborhood $V$ of $X_2$ for which the inclusion of $X_2$ into $V$ is shape $r$-connected and the pair $(h(U_0), V)$ is $(2k+2-n)$-connected.

By Observation 1 of the preceding section, there exists a small neighborhood $V$ of $X_2$ in $\text{int } h(U_0)$ for which the inclusion of $X_2$ into $V$ is shape $r$-connected. Since $r \geq 2k + 2 - n$ it follows from Observation 2 of the preceding section that to complete the argument we need only show that the inclusion of $X_2$ into $h(U_0)$ is shape $r$-connected. To this end, let $h^{-1}: h(U_0) + U_0$, $j: X_1 \rightarrow U_0$, and $k: X_2 \rightarrow h(U_0)$ be relative fundamental sequences generated by $h^{-1}: h(U_0) + U_0$ (it is here we require $h = \text{id}_M$), and the inclusions $j: X_1 \rightarrow U_0$ and $k: X_2 \rightarrow h(U_0)$. Since $h^{-1}|_W_0 = f_i|_W_0$ in $U_0$ for almost all $i$, $jf' = h^{-1}k$, and so $hjf' = k$. This shows that $k$ is an $r$-connected shape morphism, and the proof is complete.

Lemma 1 is used to maintain an inductive argument.
to prove Theorem 3, as Lemma 1 of [I-S-V] was used to prove Theorem 3 of [I-S-V]. Our hypotheses have been tailored so that the induction may be started by letting $h = \text{id}_M$ and $U_0 = M$.

Theorem 3. Let $X_1$ and $X_2$ be continua in the interior of the piecewise linear n-manifold $M$ such that for $j = 1$ or 2, $X_j$ has fundamental dimension at most $k$, $X_j$ satisfies ILC, and $\text{pro-}\pi_1(X_j)$ is stable for $0 \leq i \leq r - 1$ and satisfies the Mittag-Leffler condition for $i = r < n - 3$, where $n \geq \max(2k+2-r, k+3, 5)$. Suppose the inclusion of $X_1$ into $M$ is shape $r$-connected and that $X_1$ and $X_2$ have the same shape relative to $M$. Then $M - X_1 \approx M - X_2$.

We note that the hypothesis that the inclusion of $X_1$ into $M$ be shape $r$-connected is necessary in Theorem 3. In Counterexample 6, Chapter 8 of [Z] is shown the existence, for $m \geq 2$, of an $m$-sphere $S^m_1$ inessentially (piecewise linearly) embedded in $\text{int}(B^{2m} \times S^1)$ so that $S^m_1$ does not bound an $(m+1)$-cell in $B^{2m} \times S^1 = M$. If $S^m_2$ is a piecewise linear $m$-sphere lying in the interior of a $(2m+1)$-cell in $M$, then it is easily seen that $M - S^m_1 \not \approx M - S^m_2$.

As in Section 6 of [I-S-V] we may obtain from the proof of Theorem 3 a result on Borsuk's notion of position (see Chapter XI of [B]) which we state here as follows.

Addendum to Theorem 3. Under the hypotheses of Theorem 3, $\text{pos}(M, X_1) = \text{pos}(M, X_2)$.

4. A Piecewise Linear Embedding-up-to-Simple-Homotopy Theorem

Let $X$ be a compactum lying in the interior of the
piecewise linear manifold \( M \), and let \( K \) be a compact ANR.

If \( \{U_i\}_{i=1}^{\infty} \) is a defining sequence for \( X \), every shape morphism \( f: K \to X \) may be represented by a system map \( \bar{f} = \{f_i\}_{i=1}^{\infty} \),

\[ f_i: K \to U_i \] (cf. Chapter IX of [B]). Suppose \( Y \) is a compactum in \( M \) and \( g: Y \to K \) is a map. Since \( K \in \text{ANR} \), \( g \) extends to \( g^*: G \to K \) for some neighborhood \( G \) of \( Y \) in \( M \). Then the sequence \( \{f_i g^*\}_{i=1}^{\infty} \) satisfies condition (2) of the definition of relative fundamental sequence given on pg. 188 of [C].

If \( G \) may be so chosen that this sequence also satisfies condition (1) of that definition, then we say that \( \{f_i g^*\}_{i=1}^{\infty} \), denoted by \( fg \), is a relative fundamental sequence induced by the map \( g \) and the shape morphism \( \bar{f} \).

**Lemma 2.** Suppose \( X \) is a continuum lying in the interior of the piecewise linear \( n \)-manifold \( M \) such that \( X \) satisfies ILC and has the shape of a \( k \)-dimensional polyhedron \( K \), where \( n \geq 5 \) and \( k \leq n - 3 \). Then there exist a defining sequence \( \{U_i\}_{i=1}^{\infty} \) for \( X \), \( k \)-dimensional polyhedra \( K_i \subset U_i \), and simple homotopy equivalences \( \tilde{f}_i: K \to K_i \) such that

1. if \( f_i: K \to U_i \) is the composition \( \tilde{f}_i: K \to K_i \subset U_i \), then \( f_i \) is \((k-1)\)-connected and \( \bar{f} = \{f_i\}_{i=1}^{\infty}: K \to \{U_i\} \) is a system map which is a shape equivalence from \( K \) to \( X \),

2. if \( i \geq j > \ell \), then \( K_i \) is a strong deformation retract of \( U_j \) in \( U_\ell \), and

3. if \( i > \ell \) and \( \phi_i: K_i \to K \) is a homotopy inverse of \( \tilde{f}_i \), then \( \phi_i \) and \( f \) induce a relative shape equivalence of \( K_i \) to \( X \) in \( U_\ell \).

**Proof.** Theorem 1 yields a \((k-1)\)-connected defining
sequence for $X$ in $M$. Since $\text{Sh}(X) = \text{Sh}(K)$, there exists a shape equivalence $f_i' = \{f_i'\}_{i=1}^{\infty}$ of $K$ to $X$, where $f_i': K \to U_i$. It is easy to see that each of the maps $f_i'$ is $(k-1)$-connected. Let $f_i: K \to U_i$ be a piecewise linear general position map such that $f_i(K) \subset \text{int} U_i$ and $f_i \simeq f_i'$. Then $f_i$ is a $(k-1)$-connected map whose singular set has dimension at most $2k - n \leq k - 3 \leq n - 5$; Theorem 4.3 of [St] then applies to yield a $k$-dimensional polyhedron $K_i \subset \text{int} U_i$ such that $f_i(K) \subset K_i$ and the composition $\tilde{f}_i: K \overset{f_i}{\to} f_i(K) \hookrightarrow K_i$ is a simple homotopy equivalence. Let $\beta_{ji}: U_j \to U_i$ denote the inclusion map, where $i \geq j$. Since $\tilde{f} = \{f_i\}_{i=1}^{\infty}$ is a shape equivalence, we may assume that there are maps $g_i: U_i \to K$ such that if $j < i$, then $f_j g_i \simeq \beta_{ji}$, and if $j \leq i$, $g_j \beta_{ji} f_i \simeq \text{id}_K$ and $\beta_{ji} f_i \simeq f_j$.

We note that (1) holds. The proof of (2) is similar to the proof of Lemma 2.1 of [C-D-D]. Specifically, if $\ell < j \leq i$, then $g_j \beta_{ji} f_i \simeq \text{id}_K$, and so $f_i g_j \beta_{ji} f_i \simeq f_i$. Letting $\phi_i$ denote a homotopy inverse of $f_i$, this yields $f_i g_j \beta_{ji} f_i \phi_i \simeq \text{id}_{K_i}$. But $f_i \phi_i \simeq \text{id}_{K_i}$ in $K_i$, and so $f_i g_j | K_i \simeq \text{id}_{K_i}$. This implies that the map $f_i g_j: U_j \to K_i$ is a weak retraction of $U_j$ to $K_i$. Since $K_i \in \text{ANR}$, it follows from the Borsuk Homotopy Extension Theorem that $K_i$ is a retract of $U_j$. Since $\beta_{ji} f_i g_j \simeq f_j g_j \simeq \beta_{ji}$ and $\beta_{ji} f_i g_j(U_j) \subset K_i$, $U_j$ deforms into $K_i$ in $U_j$; since $U_j \in \text{ANR}$, a modification of Theorem 11 on pg. 31 of [Sp] shows that $K_i$ is a strong deformation retract of $U_j$ in $U_j$.

It now remains to verify (3). Let $i > \ell$ and $\phi_i: K_i \to K$ be a homotopy inverse of $f_i$. Let $G \subset \text{int} U_i$ be a regular
neighborhood of $K_i$ and $\rho: G \to K_i$ be the retraction induced by a collapse of $G$ onto $K_i$. Now, $\beta_{\ell_i} f_i \phi_i \rho: G \to U_\ell$ is homotopic to the inclusion of $G$ into $U_\ell$. Thus, if $m \geq i$,

$$\beta_{\ell_m} f_i \phi_i \rho: G \to U_\ell$$

is homotopic to the inclusion of $G$ into $U_\ell$. This implies that

$$h = \{\beta_{\ell_m} f_i \phi_i \rho, K_i, X, G\}^\infty_{m=i}$$

is a relative fundamental sequence in $U_\ell$ induced by the map $\phi_i$ and the shape morphism $f_i$. (We index $h$ by $m \geq i$ rather than $m \geq 1$, but this is an unimportant technical matter.)

If $m > i$, then $\beta_{\ell_m} f_i \phi_m: U_m \to U_i$. By the Borsuk

Homotopy Extension Theorem, it follows that there exists an extension $\psi_m: U_{i+1} \to U_i$ of $f_i g_m$ so that $\psi_m = \beta_{i,i+1}$. By

(2), there exists $\tilde{r}: U_i \to K_i$ such that $\tilde{r}$ is the final stage of a strong deformation retraction of $U_i$ to $K_i$ in $U_\ell$. Let

$r\psi_m: U_{i+1} \to U_\ell$ denote the map $\tilde{r}\psi_m$ composed with the inclusion of $K_i$ into $U_\ell$. Then

$$r\psi_m = \beta_{i,i+1}$$

and

$$r\psi_m = r\psi_{i+1}$$

in $K_i$. It follows that

$$k = \{r\psi_m, X, K_i, U_{i+1}\}^\infty_{m=i+1}$$

is a relative fundamental sequence in $U_\ell$.

It now remains to be seen that $h$ and $k$ are mutually inverse. To see that $hk = \text{id}_{K_i}$, let $V$ be a neighborhood of $K_i$, let $G'$ be a regular neighborhood of $K_i$ in $V \cap G$, let $\alpha: K_i \to U_i$ denote the inclusion, and note that if $m \geq i + 1$, then

$$(r\psi_m)(\beta_{\ell_m} f_i \phi_i \rho)|G' = rf_i g_m f_i \phi_i \rho|G' = rf_i \phi_i \rho|G' = rrf_i \phi_i \rho|G' = r\alpha f_i \phi_i \rho|G' = r\alpha \rho|G' = \rho|G' = \text{id}|G'$

in $G'$. To see that

$hk = \text{id}_X$,

let $V$ be a neighborhood of $X$ and choose $s \geq i + 1$ such that $U_s \subseteq V$. Then, if $m > s + 1$, $(\beta_{\ell_m} f_i \phi_i \rho)(r\psi_m)|U_{s+1}$

$$= \beta_{s_m} f_i \phi_i \rho\beta_i, s+1 = \beta_{s_m} f_i \phi_i \rho r\psi_{s+1}|U_{s+1} = \beta_{s_m} f_i \phi_i \rho r\psi_{s+1}$$
\[ \beta_{smm'} \beta_{f_0 g} = \beta_{smm'} \beta_{f_1 g} = \beta_{smm} g = f g = f_{s+1} + f_s = 1 \]

all homotopies occurring in \( U_s \subset V \). This completes the proof.

We are now prepared to state our embedding theorem. It generalizes the Corollary established in \([V_3]\).

**Theorem 4.** Let \( X \) be a continuum in the interior of the piecewise linear \( n \)-manifold \( M \), \( K \) be a \( k \)-dimensional polyhedron, and \( f: K \to X \) be a shape equivalence, where \( n \geq 5 \) and \( k \leq n - 3 \). Then for each neighborhood \( U \) of \( X \) in \( M \) there exists a \( k \)-dimensional polyhedron \( K' \subset U \) and a simple homotopy equivalence \( h: K \to K' \) so that the homotopy inverse of \( h \) and \( f \) induce a relative shape equivalence of \( K' \) and \( X \) in \( U \).

**Proof.** By the main result of \([V_3]\) there exists \( X' \subset \text{int} M \) such that \( X' \) satisfies ILC and such that \( X \) and \( X' \) have the same relative shape in \( U \) via relative fundamental sequences \( q: X 

\rightarrow X' \) and \( k: X' \rightarrow X \). Part (3) of Lemma 2 (as applied to \( X' \)) shows that there exist a neighborhood \( U_k \) of \( X' \) in \( U \), a \( k \)-dimensional polyhedron \( K_{k+1} \), and a simple homotopy equivalence \( f_{k+1}: K \to K_{k+1} \) so that a homotopy inverse \( \phi_{k+1} \) of \( f_{k+1} \) induces, along with \( gf \), a relative shape equivalence of \( K_{k+1} \) and \( X' \) in \( U_k \). But a relative shape equivalence in \( U_k \) is clearly a relative shape equivalence in \( U \), and so \( \phi_{k+1} \) and \( kgf \) (which is homotopic to \( f \)) induce a relative shape equivalence of \( K_{k+1} \) and \( X \) in \( U \).

If \( X \) is a compactum in the interior of the piecewise linear \( n \)-manifold \( M \), a *deleted product neighborhood* of \( X \) is a compact piecewise linear manifold neighborhood \( N \) of \( X \) in
M such that \( N - X \cong \exists N \times [0,1] \). The following generalizes Theorem 2.4 of [C-D-D] and Theorem 3 of [L]. Since deleted product neighborhoods are I-regular neighborhoods [Si] we also obtain the fact, established in [S-G-H], that I-regular neighborhoods exist for ILC embedded continua in the interior of a piecewise linear n-manifold, \( n > 5 \), which have the shape of a codimension 3 polyhedron.

**Corollary 1.** Let \( X \) be a continuum in the interior of the piecewise linear n-manifold \( M \) such that \( X \) satisfies ILC and has the shape of a \( k \)-dimensional polyhedron \( K \), where \( n > 5 \) and \( k < n - 3 \). Then \( X \) has a deleted product neighborhood in \( M \).

**Proof.** By Observation 1 of Section 2, there exists a neighborhood \( U \) of \( X \) in \( M \) such that the inclusion of \( X \) into \( U \) is shape \((k-1)\)-connected. By Theorem 4, there exists a \( k \)-dimensional polyhedron \( K' \subset U \) such that \( X \) and \( K' \) have the same relative shape in \( U \). By Theorem 3, \( U - X \cong U - K' \). A careful examination of the proof of Theorem 3 shows that in the first step of the induction we may push \( X \) into a regular neighborhood \( N \) of \( K' \) by a homeomorphism \( q: U \to U \), the remainder of the proof showing that \( N - K' \cong N - q(X) \). Then \( q^{-1}(N) \) is a deleted product neighborhood of \( X \).

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