A DIFFERENTIABLE, PERFECTLY NORMAL, NONMETRIZABLE MANIFOLD

by

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In answer to a question originally raised by Alexandroff in [A], Rudin and Zenor, using the continuum hypothesis, displayed an example of a perfectly normal, hereditarily separable, non-metrizable topological manifold \([R,Z]\). In this paper, we show that the Rudin-Zenor manifold can be constructed so that it is analytic. A key step in our construction is a modification of a theorem of Brown [B] which is interesting in its own light; namely, we show that if a differentiable manifold \(M\) has an atlas \(\{(V_i,\phi_i)\mid i \in \omega_0\}\) such that \(V_{i+1} \supset V_i\) and \(\phi_i(V_i) = \mathbb{R}^n\) for all \(i \in \omega_0\), then \(M\) is diffeomorphic to \(\mathbb{R}^n\).

The construction of the manifold follows very closely that of \([R,Z]\) and we recommend that the reader be familiar with that paper before proceeding.

Let \(X\) be a set, and let \(n\) be a fixed positive integer.

A chart is a pair \((U,\phi)\) where \(\phi: U \rightarrow \mathbb{R}^n\) is an injective function of a subset \(U\) of \(X\) onto an open subset \(\phi U\) of \(\mathbb{R}^n\).

Two charts \((U,\phi), (V,\psi)\) are compatible, if \(\psi(U \cap V)\) and \(\phi(U \cap V)\) are open subsets of \(\mathbb{R}^n\) and \(\phi^{-1}\mid_{\phi(U \cap V)}: \phi(U \cap V) \rightarrow \psi(U \cap V)\) is a diffeomorphism.

An atlas on the set \(X\) is a collection \(\{(U_j,\phi_j)\mid j \in J\}\)

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of charts such that $X = \bigcup\{U_j\mid j \in J\}$ and any two charts are compatible.

A differential structure $\mathcal{D}$ on a set $X$ is a maximal atlas. It is clear that any atlas is contained in a unique differential structure which is said to generate.

If $\mathcal{A}$ is an atlas on the set $X$, it is also clear that there is a unique topology on $X$ with the property that $\phi: U \to \phi U$ is a homeomorphism of the open set $U$ onto $\phi U$ for every chart $(U, \phi)$.

A smooth manifold is a set $X$ together with a differential structure $\mathcal{D}$ or $X$; notation: $(X, \mathcal{D})$. When there is no danger of confusion, one simply refers to the smooth manifold $X$.

Let $D(r) = \{u \in \mathbb{R}^n \mid |u| < r\}$, and let $M$ be a smooth $n$-manifold. A subset $D$ of $M$ is said to be an $n$-disk, provided there is a chart $(U, \phi)$ of $M$ such that $\phi D = D(r)$ for some positive number $r$. (This definition allows us to avoid some technicalities regarding differentiability on sets which are not open.)

If $D$ is an $n$-disk in $M$, then a map $f: M \to M$ is said to be a radial diffeomorphism in $D$, if there exist a chart $(U, \phi)$ of $M$, a positive number $\varepsilon$, and a diffeomorphism $\lambda: \mathbb{R} \to \mathbb{R}$ such that $\phi D = D(1)$, $\lambda(t) = t$ for all $t < \varepsilon$ and all $t > 1 - \varepsilon$, $f(x) = x$ for all $x \in M - D$, and $f(x) = \phi^{-1} \lambda \phi(x)$ for $x \in D$, where $\Lambda: \mathbb{R}^n \to \mathbb{R}^n$ if defined by $\Lambda(u) = \lambda(|u|)u/|u|$ if $u \neq 0$ and $\Lambda(0) = 0$. Because $f$ is the identity on $M - D$ and a diffeomorphism of Int $D$, $f: M \to M$ is in fact a diffeomorphism.

Lemma 1. If $D_1, D_2, D_3, D_4$ are $n$-disks in a smooth
manifold $M$ such that $D_i \subseteq \text{Int } D_{i+1}$ for $i = 1, 2, 3$, then there
is a diffeomorphism $f: M \rightarrow M$ such that $f(x) = x$ for
$x \in D_1 \cup (M - D_4)$ and $\text{Int } fD_2 = D_3$.

Proof. There is a radial diffeomorphism $g: M \rightarrow M$ in
$D_4$ which is the identity on a nonempty open subset $B$ of
$\text{Int } D_1$ and which maps $D_3$ into $D_1$, and there is a radial
diffeomorphism $h: M \rightarrow M$ in $D_2$ which maps $D_1$ into $V$. Put
$f = h^{-1}g^{-1}h$. If $x \in D_3$, then $h(x) \in D_3$ and $gh(x) \in D_1$ and
consequently $h^{-1}gh(x) \in \text{Int } D_2$; hence $f^{-1}D_3 \subseteq \text{Int } D_2$, and
therefore $D_3 \subseteq g(\text{Int } D_2) = \text{Int } fD_2$.

Theorem 1. If a differentiable manifold $M$ has an atlas
$\{(U_i, \phi_i) | i \in \omega_0 \}$ such that $U_i \subseteq U_{i+1}$ and $\phi_i U_i = \mathbb{R}^n$ for all
$i \in \omega_0$, then $M$ is diffeomorphic to $\mathbb{R}^n$.

Proof. Let $h_i = \phi_i^{-1}: \mathbb{R}^n \rightarrow U_i \subseteq M$. From the hypothesis
that $U_i \subseteq U_{i+1}$ for $i \in \omega_0$ it follows that there is a strictly
increasing sequence of positive integers $r_i, i \in \omega_0$ such that
$U_i(\phi_i D(r_i) | i \in \omega_0) = M$ and $h_i D(r_i) \subseteq \text{Int } h_{i+1} D(r_{i+1})$ for
$i \in \omega_0$. Put $Q_i = h_i D(r_i)$.

We assert that there exist a sequence of diffeomorphisms
$f_i: M \rightarrow M, i \in \omega_0$ and a strictly increasing sequence of
positive numbers $s_i, i \in \omega_0$ with limit $r_1$ such that $A(i): f_i$
is the identity on $M - Q_{i+1}$ and on $f_{i-1} \cdots f_1 f_0 h_i D(s_{i-1})$ and
such that $B(i): f_i \cdots f_1 f_0 h_i D(s_i) \supseteq Q_i$. To verify this
assertion assume inductively that $f_i$ and $s_i$ for $i = 0, 1, \ldots, k$
satisfy $A(i)$ and $B(i)$ for $i = 0, 1, \ldots, k$. Since $f_k \cdots f_1 f_0 Q_1$
$\subseteq \text{Int } Q_{k+1}$, there is $s_{k+1} > s_k$ such that $0 < r_i - s_{k+1} <$
$1/(k+1)$, and the lemma applies to $D_1 = f_k \cdots f_1 f_0 h_1 D(s_k)$,
$D_2 = f_k \cdots f_1 f_0 h_1 D(s_{k+1})$, $D_3 = Q_{k+1}$, and $D_4 = Q_{k+2}$ to provide
a diffeomorphism \( f_{k+1} : M \to M \) such that \( A(k+1) \) and \( B(k+1) \) hold.

To complete the proof of the Theorem, define

\[ F : \text{Int} Q_1 + M \text{ by } F(x) = \lim_{k \to \infty} F_k(x) \text{ where } F_k = f_k \cdots f_1 f_0 : M \to M. \]

Since \( F(x) = F_k(x) \) for \( x \in h_1 D(s_k) \), \( F \) is well-defined and clearly a homeomorphism onto \( M \). Since \( F \) is a diffeomorphism on each of the open sets \( \text{Int} h_1 D(s_k) \), \( k \in \omega_0 \), it is a diffeomorphism of \( \text{Int} Q_1 \) (which is diffeomorphic to \( \mathbb{R}^n \)) onto \( M \).

**Lemma 2.** Any closed smooth embedding \( \mathbb{R} \to \mathbb{R}^2 \) extends to a diffeomorphism of \( \mathbb{R}^2 \) onto itself.

**Proof.** Any closed embedding of \( \mathbb{R} \) into \( \mathbb{R}^2 \) extends to a closed embedding \( f : \mathbb{R} \times [-2,2] \to \mathbb{R}^2 \) by means of the Collaring Theorem.

Take a rectilinear triangulation \( T \) of \( \mathbb{R}^2 \setminus f(\mathbb{R} \times \{0\}) \).

The 1-simplices of \( T \) which are not contained in \( f(\mathbb{R} \times [-1,1]) \) comprise a sequence \( \{A(j) | j \in \omega\} \) with the property that for any compact set \( K \) in \( \mathbb{R}^2 \) there is an index \( j(K) \) such that \( A(j) \cap K = \emptyset \) for all \( j \geq j(K) \).

For each positive real number \( r \) define the band \( B(r) = \mathbb{R} \times [-2 + 1/r,2 - 1/r] \). We claim there is a sequence of closed embeddings \( F_n : \mathbb{R} \times [-2,2] \to \mathbb{R}^2 (n \in \omega) \) such that \( F_0 = F \) and for all \( n \in \omega \):

1. \( F_{n+1}(x) = F_n(x) \) for the points \( x \) of \( B(n) \) and
2. \( F_n(B(n)) = A(j) \) for all \( j < n \).

If such a sequence exists, define \( F : \mathbb{R} \times (-2,2) \to \mathbb{R}^2 \) by

\[ F(x) = \lim_{n \to \infty} F_n(x) ; \text{ then } F \text{ extends } f|B(1) \text{ and is a diffeomorphism onto an open set which contains every 1-simplex} \]
of the triangulation $T$ of $\mathbb{R}^2 - f(\mathbb{R} \times 0)$ and hence by simple-connectivity every point of $\mathbb{R}^2$. It follows easily that there is a diffeomorphism of $\mathbb{R}^2$ onto itself extending the original closed embedding $\mathbb{R} \to \mathbb{R}^2$.

The claim is proved by induction. Assume $F_n$ has been obtained satisfying (2).

If $A(n) \cap F_n(B(n)) = \emptyset$, it is easy to construct a diffeomorphism $f$ of $\mathbb{R}^2$ onto itself so that $g$ is the identity on $F_n(B(n))$ and $g(A(n)) \subseteq F_n(B(n+1))$. In this case, take $F_{n+1} = g^{-1}F_n$. If $A(n) \cap F_n(B(n)) \neq \emptyset$, there is a finite sequence of closed subintervals $\{C_1, C_2, \ldots, C_r\}$ so that $A(n) - \cup\{C_i | i \leq r\}$ is contained in $F_n(B(n+1))$ and so that $C_i \cap F_n(B(n)) = \emptyset$ for $i \leq r$. By a preliminary diffeomorphism, if necessary, we may assume the set of endpoints of $C_i$ is a subset of $F_n(B(n+1))$ for $i \leq r$. For each $C_j$ let $C'_j$ be an arc lying in $F_n(B(n+1)) - F_n(B(n))$ so that $C'_j \cup C'_j$ is a simple closed curve so that $C'_j \cap C_i = \emptyset$ for all $i \neq j$. Let $M = \{i \leq r | i \neq j, C_i$ is not a subset of the bounded domain of $C_j \cup C'_j\}$. For each $i \in M$, let $C''_i$ be an arc so that $C'_i \cup C'_i \cup C''_i$ is a $\theta$-curve with $C_i$ as the cross-arc such that if $i \neq j$ are in $M$, then the 2-cells bounded by $C'_i \cup C''_i$ and $C'_j \cup C''_j$ are mutually exclusive and the 2-cells bounded by $C_i \cup C'_i$ does not intersect $F_n(B(n))$. Let $M = \{i(1), i(2), \ldots, i(t)\}$. For each $i \in M$, let $h_i$ be a diffeomorphism which is the identity on the complementary domain of $C'_i \cup C''_i$ and so that $h_i$ takes the 2-cells bounded by $C'_i \cup C_i$ into $\text{Int} F_n(B(n+1))$. Let $h = h_i(1) \circ h_i(2) \circ \cdots \circ h_i(t)$ and let $F_{n+1} = h^{-1} \circ F_n$.

**Notation.** Throughout Lemma 2 and Theorem 3, we let
H = \{(0, y) | y \leq 0\}.

**Definition.** We will say that the set K is enveloped by the open set U if K ⊂ int U.

**Lemma 3.** Suppose that \{U(j)\}_{j \in \omega} is a sequence of open and connected subsets of \( \mathbb{R}^2 \), \( \text{cl} \ U(j+1) \subset U(j) \) and \( \bigcap_{j \in \omega} U(j) = \emptyset \). Suppose further that:

A. \( \{p(j)\}_{j \in \omega} \) is a sequence of points so that \( p(j) \in U(n) \) with \( \{|p(j)|\}_{j \in \omega} \) increasing and unbounded.

B. \( \{N(j)\}_{j \in \omega} \) is a family of disjoint, infinite subsets of \( \omega \).

Then there is a diffeomorphism \( g \) of \( \mathbb{R}^2 \) onto an open subset of \( \mathbb{R}^2 \) such that

1. \( \mathbb{R}^2 - g(\mathbb{R}^2) \) is \( H \).
2. each point of \( H \) is a limit point of \( \{g(p(n)) | n \in N(j)\} \) for each \( j \in \omega \).
3. \( g(U_n) \) envelopes \( H \) for each \( n \in \omega \).

**Proof.** We construct \( G \) in several steps:

**Step 1.** Let \( h_0 \) be a diffeomorphism from \( \{(x, 0) | x \in \mathbb{R}\} \) into \( \mathbb{R}^2 \) so that \( h_0(0, 0) = p(n) \) and \( h_0(\{(x, 0) | x > n\}) \subset U(n) \). Let \( h_1 \) be the extension of \( h_0 \) taking \( \mathbb{R}^2 \) onto \( \mathbb{R}^2 \) given by Lemma 2. Let \( h = h_1^{-1} \).

**Step 2.** Let \( f \) be a diffeomorphism from \( \mathbb{R}^2 \) onto \( \mathbb{R}^2 \) which leaves the set \( \{(x, 0) | x > n\} \) fixed and so that \( \{x, 0 | x > n\} \subset f(h(U(n))) \).

**Step 3.** Let \( S = \{s_i | i \in \omega \} \) be a countable dense subset of \( \mathbb{R} \). Let \( \phi \) be a diffeomorphism from \( \mathbb{R}^2 \) into \( \mathbb{R}^2 \) so that
(a) \( \phi(x,y) = (x,y') \) (i.e., \( \phi \) is fixed on its first coordinate).

(b) If \( N(j) = \{j(1), j(2), \cdots\} \), then \( \phi(j(i) + 1, 0) = (j(i) + l, s_i) \).

Thus, \( j(i) \) is the \( i \)th number in \( N(j) \) and \( \phi \circ f \circ h \) takes \( p(j(i)) \) onto \( (j(i) + l, s_i) \).

Step 4. Let \( \beta: \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( \beta(x,y) = (e^{-x}, y) \).

Step 5. Let \( \gamma: \{(x,y) | x > 0\} + \mathbb{R}^2 - H \) be defined by
\[
\gamma(x,y) = (\sqrt{x^2 + y^2} \cos(\pi/2 + 2 \arctan(y/x)), \sqrt{x^2 + y^2} \sin(\pi/2 + 2 \arctan(y/x))).
\]
Finally \( g = \gamma \circ \beta \circ \phi \circ f \circ h \) is the desired diffeomorphism.

Theorem 2. Assuming the continuum hypothesis, there is a hereditarily separable, perfectly normal, analytic manifold that is not metrizable.

Proof. We will build a \( C^\infty \)-manifold; the existence of an analytic manifold will then follow from \([K,P]\). The construction is simply a "careful" version of the construction developed in \([RZ]\). Let \( D = D(1) = \{x \in \mathbb{R}^2 | |x| \leq 1\} \) and let \( D^0 = \text{int} \ D \). Let \( \{x_\alpha | \alpha \in \omega_1\} \) be an indexing of \( D - D^0 \) (using CH). Let \( \{H_\alpha | \alpha \in \omega_1\} \) be a collection of mutually exclusive copies of \( H \). Let \( X_0 = \mathbb{R}^2 \) and let \( X_\alpha = X_0 \cup \bigcup_{\beta < \alpha} H_\beta \) and using CH, let \( \{A_\alpha | \alpha \in \omega_1\} \) be an indexing of the countable subsets of \( X \) so that \( A_\alpha \subset X_\alpha \). Let \( f_0 \) be diffeomorphism from \( \mathbb{R}^2 \) onto \( D^0 \) and let \( F \) be the function defined by
\[
f(x) = \begin{cases} 
f_0(x) & \text{if } x \in \mathbb{R}^2 \\
x_\alpha & \text{if } x \in H_\alpha
\end{cases}
\]
and let \( f_\alpha = f|_{X_\alpha} \). We will inductively construct a
differentiable structure $\mathcal{D}_\alpha$ on $X_\alpha$ such that:

1. $(X_\alpha, \mathcal{D}_\alpha)$ is diffeomorphic to $\mathbb{R}^2$: i.e. $\mathcal{D}_\alpha$ contains a chart $(X_\alpha, \phi_\alpha)$ with $\phi_\alpha(X_\alpha) = \mathbb{R}^2$.

2. If $\beta < \alpha$, then $(X_\beta, \phi_\beta) \in \mathcal{D}_\alpha$.

3. If $\gamma \leq \beta < \alpha$, $x \in H_\beta$ and $x_\beta$ is a limit point of $f(A_\alpha)$ in $D$, then $x$ is a limit point of $A_\alpha$ in $(X_\alpha, T_\alpha)$, where $T_\alpha$ is the topology on $X_\alpha$ given by $\mathcal{D}_\alpha$.

Let $\mathcal{D}_0$ be the usual differential structure on $X_0 = \mathbb{R}^2$ generated by the atlas consisting of the single chart $(X_0, \text{identity map})$.

Suppose we have $\mathcal{D}_\alpha$ satisfying (1)-(3) for all $\alpha < \lambda < \omega_1$.

**Case I.** $\lambda$ is a limit ordinal: Let $\mathcal{D}_\lambda$ be the differential structure generated by $\{(X_\theta, \mathcal{D}_\theta) \mid \theta < \lambda\}$. That $(X_\lambda, \mathcal{D}_\lambda)$ is diffeomorphic to $\mathbb{R}^2$ is given by Theorem 1.

**Case II.** $\lambda = \alpha + 1$: For each $n \in \omega$, let $U_n = f_\alpha^{-1}(D_{1/n}(x_\alpha))$, where $D_{1/n}(x_\alpha) = \{x \in D \mid d(x, x_\alpha) < 1/n\}$.

Then $\{U_n\}$ is a nested sequence of open sets in $X_\alpha$ such that $\bigcap_{n \in \omega} U_n = \phi$. Let $\{N_j\}_{j \in \omega}$ be a disjoint family of infinite subsets of $\omega$ and fix a 1-1 map $i : \alpha + 1 + \omega$. For each $n \in \omega$, choose $p_n \in U_n$ so that if $\beta \leq \alpha$ and $x_\alpha$ is a limit point of $f(A_\beta)$ in $D$, then $p_n \in A_\beta \cap U_n$ for all $n \in N_i(\beta)$.

Let $\phi$ be the diffeomorphism from $(X_\alpha, \mathcal{D}_\alpha)$ onto $\mathbb{R}^2$ given by our induction and let $g$ be the diffeomorphism given by Lemma 3 from $\mathbb{R}^2$ into $\mathbb{R}^2$ so that (1) $\mathbb{R}^2 - g(\mathbb{R}^2)$ is $H$, (2) each part of $H$ is a limit point of $\{g(\phi(p(k))) \mid k \in N_j\}$ for each $j \in \omega$, and (3) $g(U_n)$ envelopes $H$ for each $n \in \omega$. Let $\mathcal{D}_{\alpha + 1}$ be the differential structure on $X_{\alpha + 1}$ generated by the
atlas $\mathcal{D}_a \cup \{(X_{a+1}, \phi_{a+1})\}$ where $\phi_{a+1}|_{X_a} = g \circ \phi_a \cdot \phi_{a+1}|_{H_a}$ is the identification of $H_a$ with $H$.

As in [RZ], the construction of $\mathcal{D}_{a+1}$ is such that $f_{a+1}$ is continuous and our induction is complete. We will let $\mathcal{D}$ be the atlas on $X$ generated by $\bigcup_{a < \omega_1} \mathcal{D}_a$ and let $T$ be the topology on $X$ given by $\mathcal{D}$. The argument that $(X,T)$ is hereditarily separable, perfectly normal, but not Lindelöf follows exactly as in [R,Z].

**Note.** As with the Rudin-Zenor manifold, we can, using $\phi$, obtain a differentiable, perfectly normal, countably compact, hereditarily separable, non-metrizable manifold. It remains an open question if there is a complex analytic, perfectly normal, non-metrizable manifold.

**References**


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