SEQUENTIAL ORDER OF
HOMOGENEOUS AND PRODUCT SPACES

by

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In this paper we will present some examples of sequential spaces with properties related to their sequential order [1]. In section 1 we will show that homogeneous spaces may have any prescribed sequential order. In section 2 we will show that the sequential order of the sequential coreflection of a product of spaces of "small" sequential order may have "large" sequential order and will answer a question posed by Michael [4] when we give an example of a sequential \( \aleph_0 \)-space \( Z \) so that the sequential coreflection of \( Z^2 \) is not regular. Our constructions use sets of sequences as underlying sets and will be facilitated by the following notation. If \( X \) is a set, we denote finite sequences in \( X \) by \( (x_0, x_1, \cdots, x_k) \), where \( k \in \omega \) and every \( x_j \in X \). If \( \{x_0, x_1, \cdots, x_k \} \subseteq X \) and \( S \subseteq X \), we let
\[
(x_0, x_1, \cdots, x_k, S) = \{(x_0, x_1, \cdots, x_k, x_{k+1} \in S \}
\]
and let \( (x_0, x_1, \cdots, x_k, S, \cdots) \) be the set of all finite sequences in \( X \) which extend a member of \( (x_0, x_1, \cdots, x_k, S) \).

1. Homogeneous Spaces

The authors of [1] asked whether their space \( S_\omega \) is the only countable, Hausdorff, homogeneous, sequential space which is not first countable. A technique which produces many spaces with these properties was given in [3]; this technique produces spaces which, like \( S_\omega \), have sequential
order $\omega_1$. While [5] gives a non-regular example with the listed properties and has sequential order 2, the situation among regular spaces is clarified by the following.

A. Examples. For every $\alpha \leq \omega_1$ there is a countable, regular, homogeneous, weakly first countable space $X_\alpha$ with sequential order $\alpha$.

Following [1], if $A$ is a subset of a topological space $X$ then $A^0 = A$; if $\alpha = \beta+1$, then $A^\alpha = \text{the set of limits of sequences in } A^\beta$; if $\alpha$ is a limit ordinal, then $A^\alpha = \bigcup_{\beta < \alpha} A^\beta$.

We will construct our spaces by induction on $\alpha$, letting $X_\alpha$ be a countable discrete space. Suppose we have constructed $X_\beta$ for all $\beta < \alpha$.

1. $\alpha$ is not a limit ordinal.

Write $\alpha = \beta+1$; let $B$ be a wfc system [2] for $X_\beta$; pick a distinguished point $x^*$ in $X_\beta$. Let $X_\alpha$ be the set of finite sequences $(x_0, x_1, \ldots, x_{2k})$ in $X_\beta \cup \omega$ so that

- $x_j \in X_\beta$ if $j$ is even,
- $x_j \in \omega$ if $j$ is odd.

We define a wfc system $B'$ for $X_\alpha$ as follows. If $n < \omega$ and $\sigma = (x_0, x_1, \ldots, x_k) \in X_\alpha$, then

$$B'(n, \sigma) = \{\sigma\} \cup \{(x_0, \ldots, x_{k-1}, B(n, x_k) \setminus \{x_k\}, \ldots)$$

$$\cup (x_0, \ldots, x_k, \omega \setminus n, x^*, \ldots),$$

where $\omega \setminus n = \{n, n+1, n+2, \ldots\}$.

We may then show that the sets

$$\{\sigma\} \cup \{(x_0, x_1, \ldots, x_{k-1}, \cup \{x_k\}, \ldots) \cup$$

$$\bigcup_{n>n_0} (x_0, x_1, \ldots, x_k, n, V_n, \ldots)$$

hold.
where $\sigma = (x_0, x_1, \ldots, x_k) \in X_\alpha$, $U$ is a neighborhood of $x_k$ in $X_\beta$, $n_0 < \omega$, and every $V_n$ is a neighborhood of $x^*$ in $X_\beta$, make up a local neighborhood base at $\sigma$.

Inductively, $X_\alpha$ may be shown to have a base of clopen sets.

For each $\sigma = (x_0, x_1, \ldots, x_k) \in X_\alpha$ the clopen neighborhood $(x_0, x_1, \ldots, x_{k-1}, x_\beta, \ldots)$ of $\sigma$ is canonically homeomorphic to $X_\alpha = (X_\beta, \ldots)$ under a homeomorphism carrying $\sigma$ to $(x^*, \ldots)$. It follows that distinct points $\sigma$ and $\tau$ lie in disjoint clopen sets which admit a homeomorphism carrying $\sigma$ to $\tau$; hence $X_\alpha$ is homogeneous.

We wish to show that $X_\alpha$ has sequential order $\alpha$. Suppose $A \subseteq X_\alpha$ and $(x^*) \in \text{cl} A \setminus A$. Then as $\{(x^*)\} \cup (X_\beta \setminus \{x^*\}, \ldots) \cup (x^*, \omega, X_\beta, \ldots)$ is a neighborhood of $(x^*)$, we may assume that either (1) $A \subseteq (X_\beta \setminus \{x^*\}, \ldots)$, or (2) $A \setminus \{x^*, \omega, X_\beta, \ldots\}$.

(case 1) $A \subseteq (X_\beta \setminus \{x^*\}, \ldots)$. Let $\pi : (X_\beta \setminus \{x^*\}, \ldots) \to (X_\beta \setminus \{x^*\})$ be the natural "trimming" function. Since $(x^*) \in \text{cl} A$, we may use the given neighborhood base for $(x^*)$ to show that $(x^*) \in \text{cl} \pi(A)$. As $\pi(A) \subseteq (X_\beta)$ and $(X_\beta)$ is closed and homeomorphic to $X_\beta$, we deduce that $(x^*) \in [\pi(A)]_\beta$. That $(x^*) \in A^\beta$, follows from (*).

(*) For every $\gamma [\pi(A) \setminus \pi(A) \subseteq A^\gamma$.

If $\gamma = 1$ and $(x_0) \in \text{seq cl} \pi(A) \setminus \pi(A)$, then there is a sequence $(\sigma_n)_{n<\omega}$ in $A$ so that $(\pi(\sigma_n))_{n<\omega}$ converges to $(x_0)$. Recalling the definition of $B'$, we see that in fact $(\sigma_n)_{n<\omega}$ converges to $(x_0)$. Assume now that (*) holds for all $\delta < \gamma$. If $\gamma = \delta + 1$, let $(x_0) \in [\pi(A)]^\gamma \setminus \pi(A)$; we may
assume \( \langle x_\alpha \rangle \not\in [\pi(A)]^\delta \). Then there is a sequence \( (x_n)_{n<\omega} \) in 
\([\pi(A)]^\delta \setminus [\pi(A)] \) converging to \( \langle x_\alpha \rangle \), the induction hypothesis
yielding that \( \langle x_\alpha \rangle \in A^\gamma \). On the other hand, if \( \gamma \) is a
limit ordinal, then 
\([\pi(A)]^\gamma \setminus [\pi(A)] = \cup_{\delta<\gamma} [\pi(A)]^\delta \setminus [\pi(A)] \subset 
\cup_{\delta<\gamma} A^\delta = A^\gamma \). This establishes (*).

(case 2) \( A \subset \langle x^*, \omega, X_\beta, \cdots \rangle \). Let \( \pi: \langle x^*, \omega, X_\beta, \cdots \rangle \to
\langle x^*, \omega, X_\beta \rangle \) be the trimming function. Again, \( \langle x^* \rangle \in \text{cl } A \)
implies that \( \langle x^* \rangle \in \text{cl } \pi(A) \). Because \( \pi(A) \subset \langle x^*, \omega, X_\beta \rangle \cup
\{\langle x^* \rangle \} \) and because the latter set is closed and homeomorphic
to the sequential sum \([1]\) of \( \aleph_0 \) copies of \( X_\beta \) (thus has
sequential order \( \beta + 1 \)), we get \( \langle x^* \rangle \in [\pi(A)]^{\beta+1} \). We may
verify that (*) holds for \( \pi \), and thus \( \langle x^* \rangle \in A^{\beta+1} \).

Thus in either case \( \langle x^* \rangle \in A^{\beta+1} = A^\alpha \). Hence the
sequential order of \( X_\alpha \) is no greater than \( \alpha \). As \( X_\alpha \) contains
a closed copy of the sequential sum of \( \aleph_0 \) copies of \( X_\beta \), the
sequential order is precisely \( \alpha \).

2. \( \alpha \) is a limit ordinal.

Write \( \alpha = \sup \{ \beta_i : i < \omega \} \) so that the \( \beta_i \)'s are not limit
ordinals. For every \( i < \omega \), let \( X_i = X_{\beta_i} \); distinguish a
point \( x^i \) in \( X_i \), and let \( X_i^* = X_i \setminus \{x^i\} \). \( X_\alpha \) is the set of all
finite sequences \( \langle x_\alpha, x_1, \cdots, x_k \rangle \) in \( \cup_{i<\omega} X_i^* \) so that for all
\( j < k \)

if \( x_j \in X_i^* \), then \( x_{j+1} \notin X_i^* \).

Let \( B_i \) be a wfc system for \( X_i \) so that for every \( x \in X_i \) and
every \( n \leq i \), \( B_i(n, x) = X_i \). Now define a wfc system \( B \) for
\( X_\alpha \) as follows: For \( n < \omega \) and \( \sigma = (x_\alpha, x_1, \cdots, x_k) \in X_\alpha \) with
\( x_k \in X_{i_0}^* \) let
B(n, \sigma) = \{\sigma\} \cup \{x_o, x_1, \ldots, x_{k-1}, B_{i_0}(n, x_k) \setminus \{x_k\}, \ldots\}
\cup \bigcup_{i \neq i_0} \{x_o, x_1, \ldots, x_k, B_i(n, x^i) \setminus \{x^i\}, \ldots\}.

A neighborhood base at \sigma is formed by the sets of the form

\{\sigma\} \cup \{x_o, x_1, \ldots, x_{k-1}, U \setminus \{x_k\}, \ldots\} \cup
\bigcup_{i \neq i_0} \{x_o, x_1, \ldots, x_k, V_i \setminus \{x^i\}, \ldots\},

where \(U\) is a neighborhood of \(x_k\) in \(X_{i_0}\), for all \(i \neq i_0\) \(V^i\) is a neighborhood in \(X_i\) of \(x^i\), and for all but finitely many \(i\) \(V^i = x^i\). \(X_{\alpha}\) may be shown to have a base of clopen sets.

Fix \(x^o \in X^*\). Let \(\sigma = (x_o, x_1, \ldots, x_k) \in X_{\alpha}\) with \(x_k \in X_i\). We will find clopen neighborhoods of \(\langle x^o \rangle\) and \(\sigma\) that are homeomorphic under a mapping carrying \(\sigma\) to \(\langle x^o \rangle\), giving homogeneity as before. If \(i = 0\), then we can find a homeomorphism \(f\) on \(X_o\) so that \(f(x_k) = x^o\) and a clopen neighborhood \(V\) of \(x_k\) in \(X_o\) so that \(x^o \not\in U \cup f(V)\); thus \(f(V), \ldots\) and \(\langle x_o, x_1, \ldots, x_{k-1}, V, \ldots\rangle\) are the desired clopen neighborhoods. If on the other hand \(i \neq 0\), find homeomorphisms \(f_o\) on \(X_o\) and \(f_i\) on \(X_i\) so that \(f_o(x^o) = x^o\) and \(f_i(x_k) = x^i\); also find clopen neighborhoods \(V_o\) of \(x^o\) and \(V_i\) of \(x_k\) so that \(x^o \not\in f_o(V_o)\) and \(x^i \not\in V_i\). The natural map defined piecewise from the clopen neighborhood \(\{\sigma\} \cup \langle x_o, x_1, \ldots, x_{k-1}, V_i \setminus \{x_k\}, \ldots\rangle \cup \langle x_o, x_1, \ldots, x_k, V_o \setminus \{x^o\}, \ldots\rangle \cup \bigcup_{j \neq i, 0} \langle x_o, x_1, \ldots, x_j, x^*_j, \ldots\rangle\) of \(\sigma\) onto the clopen neighborhood \(\langle x^o \rangle\) \cup \langle x^o, f_i(V_i) \setminus \{x^i\}, \ldots\rangle \cup \langle f_o(V_o) \setminus \{x^i\}, \ldots\rangle \cup \bigcup_{j \neq i, 0} \langle x^o, x^*_j, \ldots\rangle\) of \(\langle x^o \rangle\) is a homeomorphism.

Suppose \(A \subset X_{\alpha}\) and \(\langle x^o \rangle \in cl A \setminus seq cl A\). Since
(x'_o) \notin \text{seq cl} A, there is an i_o < \omega so that A misses 
\bigcup_{i > i_o} (x'_o, x'^*, \ldots). We may assume that either
A \subset (X'_o\setminus\{x'_o\}, \ldots) or that A \subset (x'_o, x'^*_i, \ldots) for some i < i_o.

If A \subset (x'_o, x'^*_i, \ldots), let \pi: (x'_o, x'^*_i, \ldots) \rightarrow (x'_o, x'^*_i), be the
trimming function. Then (x'_o) \in \text{cl} \pi(A), and since
\pi(A) \subset (x'_o, x'^*_i) and (x'_o, X'^*_i) \cup \{(x'_o)\} is closed and homeo-
morphic to X_i, we have that (x'_o) \in [\pi(A)]^{\beta_i} \subset [\pi(A)]^\alpha. One
may show that \pi satisfies (*); thus (x'_o) \in A^{\alpha}. The proof
in the case A \subset (X'_o\setminus\{x'_o\}, \ldots) is similar. Hence X_\alpha has
sequential order no larger than \alpha. The subsets \{(x'_o)\} \cup
(x'_o, X'^*_i) of X_\alpha are closed and homeomorphic to X_i when
x'_o \notin X'^*_i, so the sequential order of X_\alpha is at least
\sup_{i<\omega} \beta_i = \alpha. This completes the proof.

2. Product Spaces

While the product of two sequential spaces need not be
sequential, if X and Y are sequential there is a natural
sequential space topology for the set X \times Y: decree that
a sequence \{(x_n, y_n)\}_{n \in \omega} "converges" to (x, y) if and only
if \{x_n\}_{n \in \omega} converges to x in X and \{y_n\}_{n \in \omega} converges to y
in Y; define \U to be open in X \times Y if it is sequentially
open with respect to these "convergent" sequences. This is
the sequential coreflection of the usual product topology,
denoted henceforth by \sigma(X \times Y).

Our next examples show that there is no natural bound
on the sequential order of \sigma(X \times Y) based on the sequential
orders of X and Y.
B. Examples. There are countable regular spaces $X$ and $Y$ so that $X$ is Fréchet and $Y$ is weakly first countable with sequential order 2 so that $\sigma X^2$ and $\sigma Y^2$ have sequential order $\omega_1$.

Let $X$ be the set of all finite sequences in $\omega$ of even (possibly 0) length. A set $U$ is open in $X$ if and only if for every $(n_1, n_2, \ldots, n_{2k}) \in U$ and every $i \in \omega$, there is a $j \in \omega$ so that $(n_1, n_2, \ldots, n_{2k}, i, \omega \backslash j, \ldots) \in U$.

The space $\sigma X^2$ contains a closed copy of $S_\omega$. Let $s(\phi) = (\phi, \phi)$ and for $n_1 \in \omega$ let $s(n_1) = (\langle 0, n_1 \rangle, \phi)$. Generally, $s(n_1, n_2, \ldots, n_{2k}) = (\langle 0, n_1, n_2, \ldots, n_{2k-1} \rangle, \langle n_1, n_2, \ldots, n_{2k} \rangle)$ and $s(n_1, n_2, \ldots, n_{2k+1}) = (\langle 0, n_1, n_2, \ldots, n_{2k+1} \rangle, \langle n_1, n_2, \ldots, n_{2k} \rangle)$. Observe that $(s(n_1, n_2, \ldots, n_{k+1}), n_{k+1} \in \omega)$ converges to $s(n_1, \ldots, n_k)$. We will show that these are essentially the only sequences in $S = \{s(\sigma): \sigma$ a finite sequence in $\omega\}$ converging to a point of $X^2$, hence $S$ is a sequentially closed copy of $S_\omega$ in $X^2$.

Suppose with us that there is a sequence $\sigma$ in $S$ converging to $(r_1, r_2, \ldots, r_{2k}), (s_1, s_2, \ldots, s_{2\ell}) \in X^2$ which is not eventually constant in either factor. Then there is such a $\sigma = (\langle 0, n_1^0, n_2^0, \ldots, n_{i_p}^0 \rangle, \langle n_1^p, n_2^p, \ldots, n_{j_p}^p \rangle)_{p \in \omega}$ so that

$$|i_p - j_p| = 1, \quad i_p > 2k + 1, \quad j_p > 2\ell + 2 \text{ for all } p \in \omega;$$

$(n_{2k+1}^p: p \in \omega)$ and $(n_{2\ell+1}^p: p \in \omega)$ are infinite; $(n_{2k}^p: p \in \omega)$ and $(n_{2\ell+1}^p: p \in \omega)$ are finite. Now if $j \leq 2\ell, \langle n_{j}^p \rangle_{p \in \omega}$ is eventually constant ($=s_j$), so $2k + 1 > 2\ell$; also $2\ell + 1 \neq 2k + 1$, so $2k + 1 > 2\ell + 2$. We also have that if $j \leq 2k - 1, \langle n_{j}^p \rangle_{p \in \omega}$ is eventually constant ($=r_{j+1}$), so $2\ell + 2 > 2k - 1$. With $2k \neq 2\ell + 2$, we get $2\ell + 2 > 2k + 1$, a contradiction.
So every convergent sequence in $S$ is eventually constant in one of the factors. If $\sigma$ is a sequence in $S$ which is constant in the second factor, $\sigma \setminus \langle 0, n_1, n_2, \ldots, n_{2k-1} \rangle$, $(n_1, n_2, \ldots, n_{2k}) = \langle 0, n_1, n_2, \ldots, n_{2k}, n^p_{2k+1} \rangle$, $(n_1, n_2, \ldots, n_{2k}) \in \mathcal{X}$ for some $(n_1, n_2, \ldots, n_{2k}) \in \mathcal{X}$. Likewise if $\sigma$ is constant in the first factor, $\sigma \setminus \langle 0, n_1, n_2, \ldots, n_{2k-1} \rangle, (n_1, n_2, \ldots, n_{2k-2}) = \langle (0, n_1, n_2, \ldots, n_{2k-1}), (n_1, n_2, \ldots, n_{2k}, n^p_{2k}) \rangle \in \mathcal{X}$, showing that every sequence in $S$ converging to a point in $X^2$ is eventually constant or a subsequence of one of our canonical convergent sequences, as desired.

Let $Y$ be the set of all non-void finite sequences of positive rationals with wfc system given by

$$B(m, \langle q_0, q_1, \ldots, q_k \rangle) = \langle q_0, q_1, \ldots, q_{k-1}, S_m(q_k) \rangle \cup \langle q_0, q_1, \ldots, q_k, S_m(0), \ldots \rangle$$

where $S_m(q) = \{ r \in \mathbb{Q}^+ : |r-q| < 1/m \}$.

Now let $\langle q(j) \rangle_{j<\omega}$ be a sequence in $\mathbb{Q}^+$ converging monotonically to 0 and $\langle q(j,k) \rangle_{k<\omega}$ be a sequence in $(q(j+1), q(j)) \cap \mathbb{Q}^+$ converging monotonically to $q(j)$. We will show that the set $S = \{ s(\sigma) : \sigma \text{ a finite sequence in } \omega \}$ is a closed copy of $S_\omega$ in $\sigma Y^2$, where $s(\phi) = \langle (1), (1) \rangle$, $s(n_1, n_2, \ldots, n_{2k-1}) = \langle (1, q(n_1, n_2), \ldots, q(n_{2k-3}, n_{2k-2}), q(n_{2k-1})) \rangle$, $(1+q(n_1), q(n_2, n_3), \ldots, q(n_{2k-2}, n_{2k-1}))$, $s(n_1, n_2, \ldots, n_{2k}) = \langle (1, q(n_1, n_2), \ldots, q(n_{2k-1}, n_{2k})), (1+q(n_1), q(n_2, n_3), \ldots, q(n_{2k-2}, n_{2k})) \rangle$.

We will show that a sequence $\langle s_p \rangle_{p<\omega} = \langle s(n^p_1, n^p_2, \ldots, n^p_i) \rangle_{p<\omega}$ in $S$ cannot converge to a point $\langle (r_1, r_2, \ldots, r_k) \rangle$, $\langle s_1, s_2, \ldots, s_\eta \rangle$ of $Y^2$ if for infinitely many $p < \omega$ both the first coordinate ($= s^p_1$) has length $> k$ and the second
coordinate \( \sigma_p \) has length > \( k \). For if such a sequence did converge we could, by finding a subsequence, assume that the \( \sigma_p \)'s extend \( (r_1, \ldots, r_k) \) and converge to 0 in the \( k+1 \) position, while the \( \sigma_p \)'s extend \( (s_1, \ldots, s_{k+1}) \) and converge to 0 in the \( k+1 \) position. That is, \( \{n^P_i: p < \omega\} \) is finite for all \( i < 2k - 1 \) and \( \{n^P_{2k-1}: p < \omega\} \) is infinite, while \( \{n^P_i: p < \omega\} \) is finite for all \( i < 2k \) and \( \{n^P_{2k}: p < \omega\} \) is infinite; this contradiction establishes our claim.

If \( \langle \sigma_p \rangle_{p<\omega} = \langle s(n^P_1,n^P_2,\ldots,n^P_l) \rangle_{p<\omega} \) is a sequence in \( S \) converging to \( (r_1,r_2,\ldots,r_k),(s_1,s_2,\ldots,s_{k+1}) \) so that the \( \sigma_p \)'s have length \( k \), then \( \sigma_p \) is eventually constant \( (=\langle r_1,\ldots,r_{k-1} \rangle) \) in the first \( k-1 \) positions, i.e. for appropriate \( n_1,n_2,\ldots,n_{2k-4} \) and large \( p \), \( q(n^P_{2k-5},n^P_{2k-4}) = \langle 1, q(n_1,n_2), \ldots, q(n_{2k-5},n_{2k-4}) \rangle \). Further, \( \langle \sigma_p \rangle_{p<\omega} \) converges to \( r_k \neq 0 \) in the \( k \) position, so that \( \{n^P_{2k-3}: p < \omega\} \) is eventually constant \( (=n^P_{2k-3}) \) and \( \{n^P_{2k-2}: p < \omega\} \) is infinite. Consequently \( \langle \sigma_p \rangle_{p<\omega} \) is a subsequence of \( \{s(n^P_1,n^P_2,\ldots,n^P_{2k-1},n^P_{2k-2}) \}_{n^P_{2k-2}<\omega} \).

Similarly, if \( \langle \sigma_p \rangle_{p<\omega} \) is a sequence in \( S \) converging to \( \langle r_1,r_2,\ldots,r_k),(s_1,s_2,\ldots,s_{k+1}) \) so that the \( \sigma_p \)'s have length \( l \), then \( \langle \sigma_p \rangle_{p<\omega} \) is eventually a subsequence of \( \{s(n^P_1,n^P_2,\ldots,n^P_{2l-2},n^P_{2l-1}) \}_{n^P_{2l-1}<\omega} \).

Since \( \{s(n_1,n_2,\ldots,n_j,n^P_{j+1}) \}_{n^P_{j+1}<\omega} \) converges to \( s(n_1,n_2,\ldots,n_j) \) and these are essentially the only convergent sequences in \( S \), \( S \) may be viewed as a closed copy of \( S_\omega \) in \( \sigma_\gamma^2 \).
Example. There is a regular space \( Z \) with a countable weak base \([6]\) so that \( \sigma Z^2 \) is not regular.

Let \( P \) be a countable set of irrationals which is dense in \( \mathbb{I} \), where \( \phi : P \to \mathbb{N} \) is one-to-one. Let \( Z \) be the set of all non-void sequences (finite or infinite) in \( P \) with wfc system defined as follows.

\[
\begin{align*}
B(n,\langle P_0, P_1, \ldots, P_k \rangle) &= \langle P_0, P_1, \ldots, P_{k-1}, S_n(P_k) \rangle \cup \langle P_0, P_1, \ldots, P_k, 0 \rangle, \\
B(n,\langle P_i \rangle) &= \langle P_0, P_1, \ldots, P_{n-1}, P_n, \ldots \rangle, \\
S_n(x) &= \{ y \in P : |y-x| < \frac{1}{n} \} 
\end{align*}
\]

where \( S_n(x) \) is the set of all sequences in \( P \), finite or infinite, which extend a member of \( \langle P_0, P_1, \ldots, P_k, T \rangle \). \( Z \) is regular and has a countable weak base.

For \( k \in \omega \) let

\[
\begin{align*}
W_{2k} &= \bigcup \{ B(\phi(q_k), \langle P_0, P_1, \ldots, P_k \rangle) \times B(1, \langle q_0, q_1, \ldots, q_k \rangle) : P_0, P_1, \ldots, P_k, q_0, q_1, \ldots, q_k \in P \} \\
W_{2k+1} &= \bigcup \{ B(1, \langle P_0, P_1, \ldots, P_{k+1} \rangle) \times B(\phi(P_{k+1}), \langle q_0, q_1, \ldots, q_k \rangle) : P_0, P_1, \ldots, P_{k+1}, q_0, q_1, \ldots, q_k \in P \}.
\end{align*}
\]

It is straightforward to check that a sequence converging to a member of \( W_k \) must eventually be in \( W_k \cup W_{k+1} \), and hence \( W = \bigcup_{k \in \omega} W_k \) is a sequentially open set in \( Z^2 \).

Let \( \{ P_0, q_0^\prime \} \subset P \) so that \( \phi(P_0) > (q_0^\prime)^{-1} \). We will show that every sequentially open set \( U \) in \( Z^2 \) with \( \langle P_0, q_0^\prime \rangle \in U \) contains a sequence converging to a point not in \( W \), hence that \( \sigma Z^2 \) is not regular.

Assume we have found \( p_i \) (\( i < k \)), \( q_i \) (\( i < k \)), and \( q_k^\prime \) so that

\[
\begin{align*}
1. \quad &\langle (P_0, \ldots, p_i), (q_0, \ldots, q_i) \rangle \in U \text{ if } i < k. \\
2. \quad \phi(q_i) > (p_{i+1})^{-1} \text{ and } \phi(p_i) > q_i^{-1} \text{ if } i < k.
\end{align*}
\]
3. \( \phi(p_k) > (q_k')^{-1} \).

4. \( (p_o, \cdots, p_k, q_o, \cdots, q_{k-1}, q_k') \in U \)

Because of (4) we can find an \( m \in \omega \) such that 

\[
B(m, (p_o, \cdots, p_k)) \times B(m, (q_o, \cdots, q_{k-1}, q_k')) \subseteq U;
\]

choose \( p_{k+1} \in P \) so that \( (p_o, \cdots, p_k, p_{k+1}) \in B(m, (p_o, \cdots, p_k)) \) and 

\[p_k \in P \text{ so that } (q_o, \cdots, q_k) \in B(m, (q_o, \cdots, q_{k-1}, q_k')),
\]

\[q_k^{-1} < (q_k')^{-1} < \phi(p_k), \text{ and } \phi(q_k) > (p_{k+1})^{-1}. \]

Note that 

\[
(p_o, \cdots, p_k, q_o, \cdots, q_k') \in U.
\]

Since \( (p_o, \cdots, p_k, p_{k+1}', q_o, \cdots, q_k) \in U \), there is an \( n < \omega \) so that 

\[
B(n, (p_o, \cdots, p_k, p_{k+1}')) \times B(n, (q_o, \cdots, q_k)) \subseteq U.
\]

So there is a \( q_{k+1}' \in P \) so that 

\[
q_{k+1}' \in B(n, (q_o, \cdots, q_k)) \text{ and } p_{k+1} \in P \text{ such that } (p_o, \cdots, p_{k+1}) \in \]

\[
B(n, (p_o, \cdots, p_k, p_{k+1}')), \phi(q_k) > (p_{k+1})^{-1}, \text{ and } \phi(p_{k+1}) > (q_{k+1}')^{-1}. \]

This finishes the induction.

The sequence \( \{(p_o, \cdots, p_k, q_o, \cdots, q_k): k \in \omega\} \) in \( U \) converges to \( \zeta = (p_i)_{i<\omega},(q_i)_{i<\omega} \) in \( Z^2 \). To see that 

\( \zeta \notin W \), note that if \( \zeta \in B(\phi(s_k), (r_o, r_1, \cdots, r_k)) \times B(1, (s_o, s_1, \cdots, s_k)) \) for some \( k \geq 0 \), then, since every infinite sequence in \( B(\phi(s_k), (r_o, r_1, \cdots, r_k)) \) is an extension of \( (r_o, r_1, \cdots, r_k), (p_o, \cdots, p_k) = (r_o, \cdots, r_k) \), and for the same reason \( (s_o, \cdots, s_k) = (q_o, \cdots, q_k) \). Thus 

\( \zeta \in B(\phi(q_k), (p_o, p_1, \cdots, p_k)) \times B(1, (q_o, \cdots, q_k)), \) which would mean \( \phi(q_k) < (p_{k+1})^{-1}, \) violating (2). A like argument shows that \( \zeta \) is not in any of the sets 

\[
B(1, (r_o, r_1, \cdots, r_{k+1}) \times B(\phi(r_{k+1}), (s_o, \cdots, s_k)) \text{. Thus } \zeta \notin W \text{ as claimed.}
\]

We note that \( Z \) is an \( H_0 \)-space (Theorem 1.15 in [6]), thereby answering Michael's question in [4].
References


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