ON ULTRA POWERS OF BOOLEAN ALGEBRAS

by

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0. Introduction

If \( A \) is an algebra with finitely many finitary operations and relations and if \( p \) is an ultrafilter on \( \omega \) then the reduced ultrapower \( A^\omega/p \) is also an algebra with the same operations. Keisler has shown that CH implies \( A^\omega/p \) is isomorphic to \( A^\omega/q \) for any free ultrafilters \( p, q \) on \( \omega \) when \( |A| \leq c \). In this note it is shown that if CH is false then there are two free ultrafilters \( p, q \) on \( \omega \) such that if \( (A, <) \) has arbitrarily long finite chains then \( A^\omega/p \) is not isomorphic to \( A^\omega/q \). This answers a question in [ACCH] about real-closed \( \eta_1 \)-fields. Furthermore we show that, if \( A \) is an atomless boolean algebra of cardinality at most \( c \), then each ultrafilter of \( A^\omega/p \) has a disjoint refinement, partially answering a question in [BV]. We also show that if \( B \) is the countable free boolean algebra then it is consistent that there is an ultrafilter \( p \) on \( \omega \) so that \( P(\omega)/\text{fin} \) will embed into \( B^\omega/p \) but \( B^\omega/p \) will not embed into \( P(\omega)/\text{fin} \).

1. Preliminaries

In this section the notation we use is introduced and we review some facts about ultraproducts which we will require. Our standard reference is the Comfort and Negrepontis text [CN]. Small Greek letters will denote ordinals.

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and a cardinal is an initial ordinal. If $S$ is a set and $\alpha$ is an ordinal, then $S^\alpha$ is the set of functions from $\alpha$ to $S$, $|S|$ is the cardinality of $S$ and $[S]^{<\alpha}$ is the set of subsets of $S$ of cardinality less than $\alpha$. We sometimes use $2^\alpha$ to denote cardinal exponentiation and this shall be clear from the context. If an ultrafilter $p$ on a cardinal $\alpha$ has the property that $|A| = \alpha$ for each $A \in p$ then $p$ is called a uniform ultrafilter; $U(\alpha)$ is the set of all uniform ultrafilters on $\alpha$, $\beta\alpha$ is the set of all ultrafilters on $\alpha$ and $\alpha^*$ is all free ultrafilters.

Let $\alpha$ be an infinite cardinal and let $p \in \alpha^*$, for a set $S$ the ultrapower $S^\alpha/p$ is the set of equivalence classes on $S^\alpha$ where for $s, t \in S^\alpha$, $s =^p t$ if $\{a \in \alpha : s(a) = t(a)\} \in p$. We will usually assume that when we choose $s \in S^\alpha/p$ we have in fact chosen $s \in S^\alpha$. If $L(\ , \ )$ is a binary relation on $S$ then $L(p, \ , )$ is a relation on $S^\alpha/p$ or $S^\alpha$ defined by $L(p,s,t)$ if $\{a \in \alpha : L(s(a),t(a))\} \in p$. More generally, if $p$ is any filter on $\alpha$, define $L(p,s,t)$ if $\{a \in \alpha : L(s(a),t(a))\} \in p$. If for $\gamma \in \alpha$, $S_\gamma$ is a set then the ultraproduct $\prod_{\gamma < \alpha} S_\gamma/p$ is defined similarly, as are any relations and functions. Also let $L(p,s,t,v)$ abbreviate $L(p,s,t)$ and $L(p,t,v)$. Throughout this paper $L$ will be an order (the usual order on an ordinal) and $E$ will be equality.

A function $V$ from $[\alpha]^{<\omega}$ to $P(\alpha)$ is called multiplicative if $V(H) = \{V(\{a\}) : a \in H\}$ for each $H \in [\alpha]^{<\omega}$. A filter $p$ on $\alpha$ is called $\alpha^+$-good if for each function $W$ from $[\alpha]^{<\omega}$ to $p$ there is a multiplicative function $V$ from $[\alpha]^{<\omega}$ to $p$ such that $V(H) \subseteq W(H)$ for each $H \in [\alpha]^{<\omega}$. A filter
is \( \omega \)-incomplete if it has countable many members whose intersection is empty.

A structure \((S, L)\) is \( \alpha \)-saturated if whenever fewer than \( \alpha \) sentences of the form \( \exists x \ L(s, x), \exists x \neg L(s, x), \exists x \ L(x, s) \) or \( \exists x \neg L(x, s) \) are given and any finitely many can be satisfied with a single \( x \in S \), then there is an \( x \in S \) which satisfies them all simultaneously. For example the set of rationals with the usual order is \( \omega \)-saturated but not \( \omega_1 \)-saturated. For subsets \( C, D \) of \( S \), let \( L(C, D) \) abbreviate that \( L(c, d) \) for each \( c \in C \) and \( d \in D \), in case of \( L(C, \{ d \}) \) or \( L(\{ c \}, D) \) we will omit the parentheses. For regular cardinals \( \kappa, \lambda \) we say that \((C, D)\) forms a \((\kappa, \lambda)\)-gap in \((S, L)\) if \( L(C, D) \), \( C \) is an increasing chain of order type \( \kappa \), \( D \) is a decreasing chain with order type \( \lambda \) under the reverse ordering and there is no \( x \in S \) with \( L(C, x, D) \).

Keisler introduced the notion of an \( \alpha^+ \)-good ultrafilter basically because of the following theorem. Keisler showed that assuming GCH there are \( \omega \)-incomplete \( \alpha^+ \)-good ultrafilters in \( U(\alpha) \) and Kunen later removed the GCH assumption (see [Ke], [K], [CN]).

1.1 Theorem (Keisler). \((S^\alpha/p, L(p))\) is \( \alpha^+ \)-saturated if \((S, L)\) is \( \omega \)-saturated and \( p \in U(\alpha) \) is \( \omega \)-incomplete and \( \alpha^+ \)-good.

Another result of Keisler's which we require is the following.
1.2 Theorem (Keisler). If $p \in \mathcal{U}(\alpha)$ is $\alpha^+$-good and 
{} $\{S_\gamma : \gamma < \alpha\}$ are all finite sets such that 
{} $\{\gamma : |S_\gamma| > n\} : n \in \omega \subset p$ then $|\prod_{\gamma < \alpha} S_\gamma / p| = 2^\alpha$. (Note that $p$ is 
{} $\omega$-incomplete.)

We include a proof of 1.2 because it is probably not as well known as 1.1 and to give the flavor of the use of good filters.

Proof. Let $W$ be the map from $[\alpha]^{<\omega}$ to $p$ defined by 
{} $W(H) = \{\gamma : |S_\gamma| > k\}$ where $k = |H^H|$. Suppose that 
{} $V : [\alpha]^{<\omega} \to p$ is a multiplicative function refining $W$. For 
{} each $\gamma < \alpha$, let $H_\gamma = \{\delta \in \alpha : \gamma \in V(\delta)\}$. Now define 
{} $n_\gamma = |H_\gamma|$ and note that we may assume that $S_\gamma \supset T_\gamma = n_\gamma H_\gamma$ 
{} since $V(H_\gamma) = \{V(\delta) : \delta \in H_\gamma\} \subset W(H_\gamma)$. Let $X = \prod_{\gamma < \alpha} n_\gamma / p$. 
{} Define a function $e$ from $X^\alpha$ to $\prod_{\gamma < \alpha} T_\gamma / p$ as follows: for 
{} $y \in X^\alpha$ let $e(y) \in \prod_{\gamma < \alpha} T_\gamma / p$ where $e(y)(\gamma) \in T_\gamma$ and is such 
{} that $e(y)(\gamma)(\delta) = y(\delta)(\gamma)$ for each $\delta \in H_\gamma$. Now if $y \neq z$ 
{} are both in $X^\alpha$, then for some $\delta \in \alpha - E(p,y(\delta),z(\delta))$. It 
{} follows that $\{\gamma \in \alpha : e(y)(\gamma) \neq e(z)(\gamma)\} \supset \{\gamma \in \alpha : \delta \in H_\gamma$ 
{} and $y(\delta) \neq z(\delta)\} = V(\{\delta\}) \cap \{\gamma : y(\delta)(\gamma) \neq z(\delta)(\gamma)\} \in p$ and 
{} so $e(y) \neq e(z)$. Therefore $|\prod_{\gamma < \alpha} S_\gamma / p| \geq |\prod_{\gamma < \alpha} T_\gamma / p| \geq 
{} |X^\alpha| = 2^\alpha$. The reverse inequality is trivial.

1.3 Definition. For a cardinal $\alpha$, let $\gamma \in \alpha^\alpha$ where 
{} $\gamma(\delta) = \gamma$ for $\delta \in \alpha$. For $p \in \mathcal{U}(\alpha)$, define $\kappa(i,p) = \min(\kappa : 
{} (\alpha^\alpha,L(p))$ has an $(\omega_1,\kappa)$-gap of the form $\{(\gamma : \gamma < \omega_1$ 
{} $, \{f_\delta : \delta < \kappa\}\}$ for each regular $\omega_1 \leq \alpha$. Similarly, let 
{} $b(p) = \min(\kappa : (\alpha^\alpha,L(p))$ has $(\kappa,\emptyset)$-gap). If $\alpha = \omega$, let 
{} $\kappa(\emptyset,p) = \kappa(p)$. 


1.4 Proposition. Let $p \in U(\alpha)$ be $\omega$-incomplete $\alpha^+$-good. If $(S,L)$ has increasing chains of any finite length then, for each regular $\omega_i \leq \alpha$, $\kappa(i,p)$ is the unique regular cardinal such that $(S^\alpha,L(p))$ has an $(\omega_i,\kappa)$-gap. Hence $\kappa(i,p) > \alpha$.

Proof. Let us first show that $(S^\alpha,L(p))$ has an increasing chain of order type $\alpha$. Fix $\{A_n : n \in \omega\} \subseteq p$ so that $\cap A_n = \emptyset$ and let $V$ be a multiplicative map of $[\alpha]^{<\omega}$ into $p$ with $V(H) \subseteq A_{|H|}$ for $H \in [\alpha]^{<\omega}$. For each $\delta \in \alpha$, let $H_\delta = \{\gamma \in \alpha : \delta \in V(\{\gamma\})\}$ and let $C_\delta = \{c(\delta,\gamma) : \gamma \in H_\delta\} \subseteq S$ be a chain. Define, for $\gamma \in \alpha$, $g_\gamma \in S^\alpha$ so that if $\gamma \in H_\delta$ then $g_\gamma(\delta) = c(\delta,\gamma)$. Now if $\beta < \gamma < \alpha$, then $\delta \in \alpha : L(g_\delta(\delta), g_\gamma(\delta)) \supseteq V(\{\beta,\gamma\}) \in p$. It is now clear that if $\omega_i \leq \alpha$ is regular and $\{g_\gamma : \gamma < \omega_i\} \subseteq S^\alpha$ is a chain then we may assume that $V$, $\{g_\gamma : \gamma < \omega_i\}$, $\{H_\delta : \delta \in \alpha\}$ and $\{C_\delta : \delta \in \alpha\}$ are as above. Furthermore if $h \in S^\alpha$ is such that $L(p,g_\gamma,h)$ for $\gamma < \omega_i$ then there is an $h' \in \Pi_{\delta < \alpha} C_\delta$ so that $L(p,g,h',h)$ for $\gamma < \omega_i$. Indeed, define $h'(\delta) = \max\{g_\gamma(\delta) : \gamma \in H_\delta \text{ and } L(p,g,h(\delta))\}$. Therefore, for any regular cardinal $\kappa$, if $\{h_\gamma : \gamma < \kappa\} \subseteq S^\alpha$ is such that $\{g_\gamma : \gamma < \omega_i\}$, $\{h_\gamma : \gamma < \kappa\}$ is a gap, then we may assume $\{h_\gamma : \gamma < \kappa\} \subseteq \Pi_{\delta < \alpha} C_\delta$.

Similarly in the structure $(\alpha^\alpha,L(p))$, if $\{f_\gamma : \gamma < \kappa\} \subseteq \alpha^\alpha$ is such that $\{f_\gamma : \gamma < \omega_i\}$, $\{f_\gamma : \gamma < \kappa\}$ form a gap, we may assume $f_\gamma \in \Pi_{\delta < \alpha} H_\delta$. The result now follows from the fact that $(\Pi_{\delta < \alpha} C_\delta/p,L(p))$ is isomorphic to $(\Pi_{\delta < \alpha} H_\delta/p,L(p))$.

If $B$ is a boolean algebra, then the Stone space of $B$, $S(B)$, is the space of ultrafilters of $B$ in which a set is closed and open (=clopen) precisely when it is of the form
$b^* = \{ p \in S(B) : b \in p \}$. Conversely if $X$ is a compact space with a base for the topology consisting of clopen sets (= 0-dimensional) then $CO(X)$ is the boolean algebra of clopen subsets of $X$. It is clear that $B$ is isomorphic to $CO(S(B))$ and that $X$ is homeomorphic to $S(CO(X))$. Also $B$ embeds into $CO(X)$ if and only if $X$ maps continuously onto $S(B)$. The set $\beta_\alpha$ is topologized as $S(P(\alpha))$ and both $U(\alpha)$ and $\alpha^*$ have the subspace topology. Recall that the unique countable atomless boolean algebra is equal to $CO(2^\omega)$ where $2^\omega$ is the Cantor set (i.e. $2^\omega$ has the product topology).

There is an alternate construction of an ultrapower of a boolean algebra $B$. The topological space $\alpha \times S(B)$ (where $\alpha$ has the discrete topology) has a Stone-Cech compactification $\beta(\alpha \times S(B))$. In fact, $\beta(\alpha \times S(B))$ is just the Stone space of $B^\alpha$. The map $f: \alpha \times S(B) \to \alpha$ defined by $f([\gamma] \times S(B)) = \gamma$ extends to an open map $f$ from $\beta(\alpha \times S(B))$ to $\beta_\alpha$. If we let $K^P = f^+(p)$ for $p \in U(\alpha)$ then $CO(K^P) \cong B^\alpha/p$. If $p$ is $\omega$-incomplete $\alpha^+$-good then $K^P$ is an $F_{\alpha^+}$-space in which any non-empty intersection of at most $\alpha$ many clopen sets has infinite interior (see [CN]). This is clearly not a useful way of constructing the ultrapower but the space $K^P$ is an interesting topological space and an analogous construction can be made from spaces of the form $\alpha \times Y$ where $Y$ is, for example, connected.

2. The Main Constructions

Let $(S,L)$ be an $\omega$-saturated structure with $|S| = \alpha$ and let $p,q \in U(\alpha)$ be $\omega$-incomplete $\alpha^+$-good. If $2^\alpha = \alpha^+$ then it is easily seen by 1.1 that $S^\alpha/p \cong S^\alpha/q$ because they each
have cardinality $2^\alpha$. However if $2^\alpha > \alpha^+$ it may not be the case that these ultrapowers are isomorphic. The easiest way to distinguish them would be if $\kappa(i,p) \neq \kappa(i,q)$ for some $\omega_i < \alpha$. In this section we show that there is always $p \in U(\alpha)$ so that $\kappa(i,p) = \text{cf}(2^\alpha)$ (the cofinality of $2^\alpha$) for each $\omega_i < \alpha$. Furthermore in the case of $\alpha = \omega$ we show that $\kappa(p)$ can be anything reasonable. In fact we prove the following two theorems.

2.1 Theorem. There is an $\omega$-incomplete $\alpha^+$-good ultrafilter $p$ on $\omega$ so that $\kappa(i,p) = \text{cf}(2^\alpha)$ for each regular $\omega_i < \alpha$.

2.2 Theorem. For each regular $\kappa$ with $\omega_1 < \kappa < 2^\omega$ there is a $p \in U(\omega)$ so that $\kappa(p) = \kappa$.

The reason that we are able to prove more for $\alpha = \omega$ is that every free ultrafilter on $\omega$ is $\omega^+$-good which is not the case for $\alpha > \omega$. If $(R,<,+,x)$ is the field of real numbers then $(R^\omega/p,L(p),+(p),x(p))$ with the obvious meanings is an example of a real-closed $\eta_1$-field or an H-field (see [ACCH]), for each $p$ in $U(\omega)$. From 2.2, we obtain the following answer to a question in [ACCH].

2.3 Corollary. If $2^\omega > \omega_1$ then there are non-isomorphic H-fields of cardinality $2^\omega$. These fields may all have the form $R^\omega/p$ for $p \in U(\omega)$.

This was shown to be consistent by Roitman [R] and the first sentence was shown to be consistent in [ACCH].
Recall that if $p$ is a filter on $\omega$, not necessarily maximal, and $f, g \in S^\omega$ then $L(p, f, g)$ denotes the condition \{for each $n \in \omega$, $f(n) < g(n)$\} $\in p$. If $p$ is the cofinite filter on $\omega$ then we use $f <^* g$ rather than $L(p, f, g)$. Recall that $b = \min\{|F| : F \subset \omega^\omega$ and there is no $g \in \omega^\omega$ such that $f <^* g$ for all $f \in F\}$ and $d = \min\{|F| : F \subset \omega^\omega$ and for each $g \in \omega^\omega$ there is an $f \in F$ with $g <^* f\}$. It is easily seen that, for any $p \in U(\omega)$, $b \leq b(p) \leq d$ and since it is consistent that $b = d = \kappa$ for any regular $\kappa$ with $\omega_1 \leq \kappa < 2^\omega$, $b(p)$ cannot take the place of $\kappa(p)$ in 2.2. On the other hand it is a result of Rothberger that $\min\{\kappa : P(\omega)/\text{fin}$ has an $(\omega, \kappa)$-gap\} and it is easily shown that $b = \min\{\kappa : (\omega^\omega, <^*)$ has an $(\omega, \kappa)$-gap\} hence it is somewhat surprising that $\kappa(p)$ need not equal $b$ or $b(p)$. However for $P$-points in $U(\omega)$ $\kappa(p) \geq b$ (I do not know if $\kappa(p) = b(p)$). A point $p \in S(B)$, for a boolean algebra $B$, is a $P_\alpha$-point if $p$ is an $\alpha$-complete filter, a $P$-point is a $P_{\omega_1}$-point (i.e. if $A \in [p]^\omega$ then there is a $b \in p$ with $b < a$ for each $a \in A$).

2.4 Proposition. If $p \in U(\omega)$ is a $P$-point then $b \leq \kappa(p) \leq d$.

Proof. If $g \in \omega^\omega$ and $L(p, n, g)$ for each $n \in \omega$, then there is an $f \in \omega^\omega$ such that $E(p, g, f)$ while $f^+(n)$ is finite for each $n \in \omega$. Now let \{\{q_\alpha : \alpha < \kappa(p)\} \subset \omega^\omega$ be chosen so that $|q^+(\alpha)(n)| < \omega$ for each $n \in \omega$ and \{\{n : n \in \omega\}, \{q_\alpha : \alpha < \kappa(p)\}\} forms a gap in $(\omega^\omega, L(p))$. For each $\alpha < \kappa(p)$ and $n \in \omega$ define $f^\alpha(n) = \min\{k : q^\alpha(j) > n$ for $j > k\}$. We show that \{\{f^\alpha : \alpha < \kappa(p)\}$ is unbounded in $(\omega^\omega, <^*)$. Indeed suppose that $f \in \omega^\omega$ is strictly increasing and $f^\alpha <^* f$ for
\(a < \kappa(p)\). Define \(g(k) = \max(n: f(n) \leq k)\) for \(k \in \omega\). Let \(a < \kappa(p)\) and choose \(m \in \omega\) so that \(f(n) > f_a(n)\) for \(n > m\).

Now let \(j > f(m)\) and let \(g(j) = n\), hence \(f_a(n) < f(n) < j\) which means that \(g_a(j) > n\). Therefore \(g < * g_a\) for all \(a < \kappa(p)\), which is a contradiction; and so \(\kappa(p) > b\). Now let \(H \subset \omega^\omega\) be increasing functions with \(|H| = d\) so that for each \(f \in \omega^\omega\) there is an \(h \in H\) with \(f < * h\). For each \(f \in H\), \(|\{a < \kappa(p): f_a < * f\}| < \kappa(p)\) since otherwise we could define \(g\) as above and have \(g < * g_a\) for \(a < \kappa(p)\). Therefore, since, for each \(a < \kappa(p)\), there is an \(h \in H\) with \(f_a < * h\), \(\kappa(p) \leq |H|\).

Before we can give the proofs of 2.1 and 2.2 we need some preliminary results.

2.5 Definition. Let \(F \subset \alpha^\alpha\) and let \(p\) be a filter on \(\alpha\). \(F\) is of large oscillation mod \(p\) if for any \(n < \omega\), \(\{f_1, \ldots, f_n\} \subset F\), \((\gamma_1, \ldots, \gamma_n) \in \alpha^n\) and \(A \in p\) the set \(A \cap \bigcap \{f_i^+(\gamma_n): 1 \leq i \leq n\}\) is not empty.

The above definition and the following result are in [EK].

2.6 Theorem. There is a set \(F \subset \alpha^\alpha\) of cardinality \(2^\alpha\) such that \(F\) is of large oscillation mod \(p\) where \(p = \{A \subset \alpha: |\alpha \setminus A| < \alpha\}\).

Kunen constructed \(\alpha^+\)-good ultrafilters on \(\alpha\) using the following idea.

2.7 Lemma. Suppose that \(p\) is a filter on \(\alpha\), \(F \subset \alpha^\alpha\) is of large oscillation mod \(p\), \(W\) is a function from \([\alpha]^{<\omega}\) into
p and A is a subset of \( \alpha \). There is a filter \( p' \supseteq p \) and \( F' \subseteq F \) and a multiplicative function \( V \) from \( [\alpha]^{<\omega} \) into \( p' \) so that \( V \) refines \( W \), \( |F \setminus F'| < \omega \), either \( A \) or \( \alpha \setminus A \) is in \( p' \) and \( F' \) is of large oscillation mod \( p' \).

Proof. We first find \( V \). Let \( \{ H : \gamma < \alpha \} \) be a listing of \( [\alpha]^{<\omega} \) and let \( f_0 \in F \) be arbitrary. For each \( H \in [\alpha]^{<\omega} \), let \( W'(H) = \cap W(J) : J \subseteq H \}, \) and define \( V(H) = \cup \{ f_0^+(\gamma) \cap W'(H) : H \subseteq H_\gamma \}. \) For each \( \delta \in H \) and \( \gamma \) with \( H \subseteq H_\gamma \), \( V(\{ \delta \}) \cap f_0^+(\gamma) = W'(H_\gamma) \) and for \( \gamma \) with \( H_\gamma \neq \emptyset \) there is a \( \delta \in H \) with \( V(\{ \delta \}) \cap f_0^+(\gamma) = \emptyset \). It follows that \( V \) is multiplicative. Let \( p_0 \) be the filter generated by \( p \cup \{ V(\{ \delta \}) : \delta < \alpha \} \); \( p_0 \) is a filter since for \( D \in p, \gamma < \alpha \ D \cap V(H_\gamma) \supseteq D \cap W'(H_\gamma) \cap f_0^+(\gamma) \neq \emptyset \). It is routine to check that \( F \setminus \{ f_0 \} \) is of large oscillation mod \( p_0 \). If \( F \setminus \{ f_0 \} \) is of large oscillation mod the filter generated by \( p_0 \cup \{ A \} \) then let these be \( F' \) and \( p' \) respectively. Otherwise there are \( f_1, \ldots, f_n \in F \setminus \{ f_0 \}, (\gamma_1, \ldots, \gamma_n) \in \alpha^n \) and \( D \in p_0 \) with \( D \cap A \cap \cap f_i^+(\gamma_i) : i = 1, \ldots, n) = \emptyset \). In this case we let \( F' = F \setminus \{ f_0, f_1, \ldots, f_n \} \) and let \( p' \) be the filter generated by \( p_0 \cup \{ f_i^+(\gamma_i) : i = 1, \ldots, n \} \).

The construction of an \( \omega \)-incomplete \( \alpha^+ \)-good ultrafilter is then just an induction of length \( 2^\alpha \) using 2.7 and being sure to introduce enough multiplicative functions and to make sure it is maximal. In order to prove 2.1 we simply add a few steps to the induction according to 2.8.

2.8 Lemma. If \( p \) and \( F \) are as in 2.7, \( \omega_i \leq \alpha \) and 
\( H = \{ h \in \alpha^\alpha : L(p, \gamma, h) \) for all \( \gamma < \omega_i \} \) then there is a
filter $p'$ and a function $f \in F$ so that $L(p, \gamma, f, h)$ for 
$\gamma < \omega_1$, $h \in H$ and $F \setminus \{f\}$ is of large oscillation mod $p'$.

Proof. Let $f \in F$ be arbitrary and let $p'$ be the 
filter generated by $p \cup \{u(f^+(\delta) \cap h^+((\delta, \omega_1)) : \gamma < \delta < \omega_1) : \gamma < \omega_1$ and $h \in H\}$. We show that $F \setminus \{f\}$ is of large oscillation mod $p'$. Indeed suppose that $f_1, \ldots, f_n \in F \setminus \{f\},$
$(\gamma_1, \ldots, \gamma_n) \in \alpha^n$, $A \in p$, $\gamma < \omega_1$ and $h \in H$ (note that $H$ is 
closed under finite meets). Let $\gamma < \delta < \omega_1$, then
$A \cap h^+((\delta, \omega_1)) = A' \in p$ since $L(p, \gamma, h)$. Therefore
$A' \cap f^+(\delta) \cap \{f_j^+(\gamma_j) : j = 1, \ldots, n\} \neq \emptyset$. It is clear that,
for $\gamma < \omega_1$, $L(p, \gamma, f)$ and, for $h \in H$, 
$\{j < a : f(j) < h(j)\} \supset
\{f^+(\delta) \cap h^+((\delta, \omega_1)) : \delta < \omega_1\} \in p'$.

Proof of Theorem 2.1. Starting with a family $F$ given
in 2.6 perform an induction of length $2^\alpha$ to construct a
chain of filters $\{p_\delta : \delta < 2^\alpha\}$ using, for instance, 2.8 when
$\text{cf}(\delta) = \omega_1$ and 2.7 otherwise. To see that, for $\omega_1 \leq \alpha$ with
$\omega_1$ regular, $\kappa(i, p) \geq \text{cf}(2^\alpha)$ observe that if $H \subseteq \alpha^\alpha$,
$|H| < \text{cf}(2^\alpha)$ and $L(p, \gamma, f)$ for $\gamma < \omega_1$ and $h \in H$ then there
is some $\delta < 2^\alpha$ with $\text{cf}(\delta) = \omega_1$ such that $L(p_\delta, \gamma, h)$ for
$\gamma < \omega_1$ and $h \in H$. Therefore by 2.8, there is an $f \in \alpha^\alpha$ with
$L(p_{\delta+1}, \gamma, f, h)$ for $\gamma < \omega_1$, $h \in H$. Also, if $D \subseteq 2^\alpha$ is cofinal
with $\text{cf}(\delta) = \omega_1$ for $\delta \in D$, then there are $f_\delta$, $\delta \in D$, so
that if $L(p, \gamma, h)$ then $L(p_\delta, \gamma, h)$ for some $\delta \in D$ and so
$L(p, \gamma, f_\delta, h)$.

Proof of Theorem 2.2. In this case $\alpha = \omega$ and so we do
not have to worry about making the filter $\alpha^+$-good. Let $\kappa$
be any regular cardinal with $\omega_1 < \kappa \leq 2^\omega$ and let $F \subseteq \omega^\omega$ be
of large oscillation mod the cofinite filter with $|F| = \kappa$.

For each $f \in F$, let $g_f$ be the map from $U(\omega)$ onto the ordinal space $\omega + 1$ defined by $g_f^+(n) = [f^+(n)]^*$ and $g_f^-(\omega) = U(\omega) \setminus \bigcup \{g_f^+(n) : n \in \omega\}$. Now let $G$ be the map from $U(\omega)$ onto $(\omega + 1)^F$ which is just the product of the $g_f$'s, $f \in F$.

Finally, using a Zorn's Lemma argument, we find a closed set $K \subset U(\omega)$ so that $G$ maps $K$ onto $(\omega + 1)^F$ but no proper closed subset of $K$ maps onto $(\omega + 1)^F$. Let $p_\beta = \{A \subset \omega : K \subset A^*\}$ and note that $F$ is of large oscillation mod $p_\beta$ since $K \cap \cap \{f_i^+(n_i)^* : i = 1,\ldots,n\} \neq \emptyset$ for all $\{f_1,\ldots,f_n\} \subset F$ and $n_i \in \omega$. The following Fact is the key to the whole proof. Let $F = \{f_\alpha : \alpha < \kappa\}$ and let $X = (\omega + 1)^F$.

**Fact 1.** If $A \subset \omega$ then there is a countable set, $\text{supp}(A) \subset \kappa$ such that if $x \in G(A^* \cap K)$ and $y \in X$ with $y(f_\alpha) = x(f_\alpha)$ for $\alpha \in \text{supp}(A)$ then $y \in G(A^* \cap K)$, and $\text{supp}(A)$ is minimal with respect to this property.

**Proof of Fact 1.** Let $S = \bigcup \{\omega^H : H \in [F]^{<\omega}\}$ and for $s \in S$ let $[s]$ be the clopen subset of $X$ given by $[s] = \{x \in X : s \subset x\}$. Recall that each non-empty open subset of $X$ contains an element of $S' = \{[s] : s \in S\}$ and that any set of pairwise disjoint members of $S'$ is countable. Now, for $A \subset \omega$, choose $T \in [S']^{<\omega}$ so that $T' = \{[t] : t \in T\}$ is a maximal collection of pairwise disjoint clopen subsets of $G(A^* \cap K)$. Clearly, $\bigcup T'$ is dense in $X \setminus G(K \setminus A^*)$. Therefore $G(G^+(\bigcup T')) \cap G(K \setminus A^*) = X$ and since $(G^+(\bigcup T')) \cap K \setminus A^*$ is closed, it follows that $G(A^* \cap K) = \overline{\bigcup T'}$. Let $\text{supp}_T(A) = \{\alpha : f_\alpha \in U(t : t \in T)\}$. Now since $x \in G(A^* \cap K)$ if and only if $x \in \overline{\bigcup T'}$ the proof of Fact 1 is complete if we can find a
minimal $\text{ supp}(A)$. Indeed $\text{ supp}(A) = \{ \alpha : \exists s \in S \text{ and } n < \omega \text{ such that } [s] \not\in G(A^* \cap K) \text{ and } [s \cup \{f_{\alpha}, n\}] \subseteq G(A^* \cap K) \}$. By definition $[\mathfrak{t}_s \text{ supp}(A)] \subseteq G(A^* \cap K)$ for each $t \in T$, so it suffices to show that $\text{ supp}(A) \subseteq \text{ supp}_T(A)$. Suppose $\alpha \in \text{ supp}(A)$ and $s, n$ exhibit this fact. Then let $y \in [s] \setminus G(A^* \cap K)$ and let $x(f_\beta) = y(f_\beta)$ for $\beta \neq \alpha$ and $x(f_\alpha) = n$. Since $[s \cup \{f_\alpha, n\}] \subseteq G(A^* \cap K), x \in G(A^* \cap K)$. Since $\text{ supp}_T(A)$ has the first property stated in Fact 1 it follows that $\text{ supp}(A) \subseteq \text{ supp}_T(A)$ and we are done.

We define a chain of filters $\{p_\alpha : \alpha < \kappa \}$ so that if $\text{ supp}(A) \subseteq \alpha$ then $A$ or $\omega \setminus A$ is in $p_{\alpha+1}$, if $A \in p_\alpha$ then $\text{ supp}(A) \subseteq \alpha$ and $\{f_\delta : \delta \geq \alpha \}$ is of large oscillation mod $p_\alpha$. Suppose $\alpha < \kappa$ and we have defined $\{p_\gamma : \gamma < \alpha \}$. If $\alpha$ is a limit then let $p_\alpha = \bigcup\{p_\gamma : \gamma < \alpha \}$. Now suppose that $\alpha = \gamma + 1$ and let $H_\gamma = \{h \in \omega^\omega : L(p_\gamma, n, h) \text{ for } n < \omega \}$. Just as in 2.8, let $p'_\gamma$ be the filter generated by $p_\gamma \cup \{\varphi(p_\gamma, n, h) : n > m \in \omega, h \in H_\gamma \}$. Extend $p'_\gamma$ to a filter $p_\alpha$ maximal with respect to the property that $A \in p_\alpha$ implies $\text{ supp}(A) \subseteq \alpha$.

Let us check that $\{f_\delta : \delta \geq \alpha \}$ is of large oscillation mod $p_\alpha$. First of all, by the minimality of $\text{ supp}(A)$ for $A \subseteq \omega$, it is clear that $\text{ supp}(A) \subseteq \alpha$ for $A \in p'_\gamma$. Now if $A \in p'_{\alpha'}$, then $\text{ supp}(A) \subseteq \alpha$ and also $G(K \cap A^*) \neq \emptyset$ because $p'_{\alpha'} \supseteq p_\alpha$. Choose $x \in G(K \cap A^*)$ and let $\{\delta_i : i = 1, \cdots, n\} \subseteq \kappa \setminus \gamma$ and $n_i \in \omega i = i, \cdots, n$. Let $y \in X$ be defined so that $\gamma(f_{\delta_i}) = n_i$ for $i = 1, \cdots, n$ and $\gamma(f_{\gamma}) = x(f_{\gamma})$ for $\gamma < \alpha$. By Fact 1, $y \in G(K \cap A^*)$ and clearly $y \in G(\cap (f_{\delta_i}^+(n_i)))$: 
Therefore \( A^* \cap \mathcal{N}\{f^*_{\delta_1}(n_i)^* : i = 1, \ldots, n\} \neq \emptyset \) since \( \mathcal{N}\{f^*_{\delta_1}(n_i)^* : i = 1, \ldots, n\} \supseteq G^*(y) \).

Finally we must show that if \( p = \bigcup \{ p_{\alpha} : \alpha < \kappa \} \) then \( \kappa(p) = \kappa \). Indeed, let \( H \subseteq \omega^\omega \) with \( |H| < \kappa \) and suppose that \( L(p, n, h) \) for each \( n \in \omega \) and \( h \in H \). Let \( \gamma < \kappa \) be large enough so that for each \( n \in \omega \), \( h \in H \), \( \text{supp}(h^+(n, \omega)) \subseteq \gamma \). Therefore \( H \subseteq H_\gamma \) and by our construction \( L(p, n, f^+_{\gamma+1}, h) \) for each \( n \in \omega \) and \( h \in H \). Therefore \( \kappa(p) = \kappa \).

As mentioned above Roitman proved that 2.2 holds consistently. In fact her techniques can be used to prove much more; it is consistent that \( B^\omega/p \) can be \( \mathfrak{c} \)-saturated providing that \( B = \text{CO}(2^\omega) \).

2.9 Theorem [R]. If \( M \) is a model obtained by adding \( \omega_2 \) Cohen reals to a model of \( 2^\omega = \omega_1 \), \( 2^{\omega_1} = \omega_2 \), then there is a \( p \in U(\omega) \) such that \( \text{CO}(2^\omega)^\omega/p \) is \( \omega_2 \)-saturated.

This is also a theorem of MA (Martin's Axiom) and even \( \mathcal{P}(\mathfrak{c}) \). \( \mathcal{P}(\mathfrak{c}) \) holds if for each free filter \( p \) on \( \omega \) with \( |p| < \mathfrak{c} \) there is an infinite \( A \subseteq \omega \) so that \( |A \setminus D| < \omega \) for \( D \in p \).

2.10 Theorem. \( \mathcal{P}(\mathfrak{c}) \) There is a point \( p \in U(\omega) \) so that \( \text{CO}(2^\omega)^\omega/p \) is \( \mathfrak{c} \)-saturated. Furthermore \( p \) can be chosen to be a \( \mathcal{P}(\mathfrak{c}) \)-point.

Proof. \( \mathcal{P}(\mathfrak{c}) \) implies that \( 2^\kappa = \mathfrak{c} \) for each \( \kappa < \mathfrak{c} \) and so we choose a listing \( \{(F_\gamma, G_\gamma) : \gamma < \mathfrak{c}\} \) of all pairs of subsets of size less than \( \mathfrak{c} \) of \( \text{CO}(2^\omega)^\omega \) so that each pair appears \( \mathfrak{c} \) times. Construct a chain of filters on \( \omega \),
\{p_\gamma : \gamma < \aleph_1\}$, so that $|p_\gamma| \leq \omega \cdot |\gamma|$ as follows. We set $p_\emptyset = \emptyset$, $p_1 = \text{cofinite}$. At limits we take unions and at successor steps we ensure that if $F \cup G$ is a chain under $L(p)$ and $L(p,F,G)$ then there is an $h \in B^\omega$ with $L(p_{\gamma + 1}, F, G)$ where $B = \{b_m : m \in \omega\} = CO(2^\omega) \setminus \{\emptyset\}$. Indeed, for $A \in p_\gamma$, $f \in F$, and $g \in G$, let $A_{f,g} = \{(k,m) : k \in A, f(k) < b_m < g(k)\}$. If $L(p,F,G)$, then $q_\gamma = \{A_{f,g} : A \in p, f \in F, g \in G\}$ is a filter base of cardinality less than $\aleph_1$. By $P(\omega)$, we choose $C \subseteq \omega \times \omega$ such that $|C \setminus A_{f,g}| < \omega$ for each $A_{f,g} \in q_\gamma$. Now since $C$ is infinite and $p_\gamma$ contains the cofinite filter, $D = \{k : C \cap \{k\} \times \omega \neq \emptyset\}$ is infinite. Define $h \in B^\omega$ so that, for $k \in D$, $h(k) = b_m$ implies $m \in C$.

Now if we let $p_{\gamma + 1}$ be the filter generated by $p_\gamma \cup \{D\}$ then $\{k \in D : f(k) \notin h(k) \text{ or } h(k) \notin g(k)\} \subseteq \{k : C \setminus A_{f,g} \cap \{k\} \times \omega \neq \emptyset\}$ and so is finite. Also $D \setminus A$ is finite for each $A \in p_\gamma$, hence $p = \cup p_\gamma$ is a $P_{\omega_1}$-point. Now $B^\omega/p$ has no $(\kappa,\lambda)$-gaps for $\kappa,\lambda < \aleph_1$ and by a result in $[D]$ this ensures that it is $\aleph_1$-saturated.

### 3. Applications to Boolean Algebras and Topology

If $B$ is an atomless boolean algebra and $p \in U(\omega)$, it follows from 1.1 that $B^\omega/p$ is an $\omega_1$-saturated boolean algebra. It is well known that $P(\omega)/\text{fin}$ is $\omega_1$-saturated and so it is natural to be interested in determining which properties $B^\omega/p$ and $P(\omega)/\text{fin}$ share and which they need not. In particular Balcar and Vojtas showed that each ultrafilter of $P(\omega)/\text{fin}$ has a disjoint refinement and asked for which other algebras is this true. Also van Douwen showed that this and some other properties of $P(\omega)/\text{fin}$ are shared...
by those $\omega_1$-saturated boolean algebras of cardinality $\mathfrak{c}$ whose Stone spaces map onto \( U(\omega) \) by an open map.

A point $x$ in a space $X$ is called a $\kappa$-point for a cardinal $\kappa$ if there are $\kappa$ disjoint open subsets of $X$ such that $x$ is in the closure of each. If $X = S(B)$ where $B$ is an $\alpha^+$-saturated boolean algebra and $\kappa = 2^\alpha$, then this is equivalent to the corresponding ultrafilter of $B$ having a disjoint refinement (that is, there is a function $f$ from $p S(B)$ to $B\setminus\{0\}$ such that $f(b) < b$ and $f(b) \land f(c) = 0$ for $b, c \in p$). A subset $\{b(i,j): (i,j) \in I \times J\}$ of $B$ is called an $I \times J$-independent matrix if $b(i,j) \land b(i,j') = 0$ and $\land\{b(i,f(i)): i \in I'\} \neq 0$ for any $i \in I' \in [I]^{<\omega}$, $f \in J^{I'}$ and $j \neq j' \in J$. $B$ has an $I \times J$-independent matrix if and only if $S(B)$ maps onto $(D(J) + 1)^I$ where $D(J) + 1$ has the product topology and $D(J) + 1$ is the one point compactification of the discrete space $J$. Kunen introduced independent matrices in [K2], he showed that $P(\omega)/\text{fin}$ has a $2^\omega \times 2^\omega$-independent matrix and used this to construct $2^\omega$-OK points. As mentioned above Balcar and Vojtas [BV] showed that every point of $U(\omega)$ is a $2^\omega$-point.

3.1 Theorem [vD]. Let $B$ be an $\omega_1$-saturated boolean algebra with $|B| = 2^\omega$ such that $S(B)$ maps onto $U(\omega)$ by an open map. (For example see the end of section 1).

(0) $S(B)$ has P-points if and only if $U(\omega)$ has P-points.

(1) $B$ has a $2^\omega \times 2^\omega$-independent matrix.

(2) Every point of $S(B)$ is a $2^\omega$-point.
(3) If \( P(\omega)/\text{fin} \) has an \((\omega,\lambda)\)-gap then so does \( B \).

(In particular \( B \) has an \((\omega,b)\)-gap and it is consistent that \( b < \lambda \)).

Now let \( \alpha \) be an infinite cardinal and let \( B \) be any atomless boolean algebra with \( |B| \leq 2^\alpha \). Also let \( p \) be an \( \omega \)-incomplete \( \alpha^+ \)-good ultrafilter on \( \alpha \).

3.2 Theorem. (0) \( S(B^\alpha/p) \) has a dense set of \( P^\alpha \)-points.

(1) \( B^\alpha/p \) has a \( 2^\alpha \times 2^\alpha \)-independent matrix.

(2) Each point of \( S(B^\alpha/p) \) is a \( 2^\alpha \)-point.

(3) \( B^\alpha/p \) has an \((\omega_1,\kappa)\)-gap if and only if \( \kappa = \kappa(i,p) \)

for each regular \( \omega_1 < \alpha \).

3.2 (0) Proof. Let \( f \in (B\setminus\{0\})^\alpha \) and for each \( \gamma < \alpha \) choose \( y_\gamma \in S(B) \) so that \( f(\gamma) \in y_\gamma \). We show that \( x = \{ g \in B^\alpha/p : g(\gamma) \in y_\gamma \text{ for } \gamma \in \alpha \} \) is a \( P^\alpha \)-point of \( S(B^\alpha/p) \). Indeed, let \( \{ g_\delta : \delta < \alpha \} \subseteq x \) and \( \{ A_n : n \in \omega \} \subseteq p \) so that \( \cap A_n = \emptyset \). Define \( W: [\alpha]^{<\omega} \to p \) by \( W(H) = A|_H \cap \{ \gamma < \alpha : g_\delta(\gamma) \in y_\gamma \text{ for } \delta \in H \} \). Now let \( V: [\alpha]^{<\omega} \to p \) be a multiplicative function refining \( W \). As usual, for each \( \gamma \in \alpha \), \( H_\gamma = \{ \delta \in \alpha : \gamma \in V(\{\delta\}) \} \) is finite. Also, since \( V(H_\delta) \subseteq W(H_\delta) \) and \( B \) is atomless we may choose \( g(\gamma) \in y_\gamma \) so that \( g(\gamma) < g_\delta(\gamma) \) for \( \delta \in H_\gamma \). It follows that \( g \in x \) and that \( L(p,g,g_\delta) \) for each \( \delta < \alpha \).

3.2 (1) Proof. Since \( B \) is atomless we may choose \( \{ b(n,m) : n,m \in \omega \} \subseteq B \) to be an \( \omega \times \omega \)-independent matrix (i.e. \( S(B) \) maps onto \((\omega + 1)^\omega \)). For each \( f,g \in \omega^\alpha/p \) define \( a_{fg} \in B^\alpha \) by \( a_{fg}(\gamma) = b(f(\gamma),g(\gamma)) \). We verify that
\{a_{fg}: f, g \in \omega^\alpha/p\} is an independent matrix. Indeed, if 
f, g, h \in \omega^\alpha with L(p, g, h) then \{\gamma \in \alpha: a_{fg}(\gamma) \land a_{fh}(\gamma) = 0\} = 
\{\gamma \in \alpha: b(f(\gamma), g(\gamma)) \land b(f(\gamma), h(\gamma)) = 0\} = \{\gamma \in \alpha: g(\gamma) \neq 
h(\gamma)\} \in p. Similarly if F is a finite subset of \omega^\alpha/p and 
G is a function from F into \omega^\alpha/p then \{\gamma \in \alpha: \land\{a_{fg}, G(f)(\gamma): 
f \in F\} \neq 0\} \supset \{\gamma \in \alpha: \land\{b(f(\gamma), G(f)(\gamma)): f \in F \neq 0\}\supset 
\{\gamma \in \alpha: |\{f(\gamma): f \in F\}| = |F|\} \in p.

Before we prove 3.2(2) we prove a result which is 
proven about P(\omega)/fin in [BV] although it is not stated 
explicitly.

3.3 Lemma. If \lambda \leq \alpha and \{a_\eta: \eta < \lambda\} \subset B^\alpha/p with 
a_\eta \land a_\xi = 0 for \eta < \xi < \lambda then the set C = \{b \in B^\alpha/p: 
\{n: b \land a_\eta \neq 0\}\} is infinite has a disjoint refinement.

Proof. Let \{A_m: m \in \omega\} \subset p with \bigcap A_m = \emptyset and for 
H \in [\lambda]^\omega define W(H) = \{\gamma \in \alpha: a_\eta(\gamma) \neq 0 and a_\eta(\gamma) \land a_\xi(\gamma) 
= 0 for \eta \neq \xi and \eta, \xi \in H\} \cap A_{|H|}. Let V be a multiplicative 
map from [\lambda]^\omega to p which refines W. Let C = \{c_\delta: 
\delta \in 2^\alpha\} and define \textbf{I}_\delta = \{\eta \in \lambda: L(p, 0, c_\delta \land a_\eta)\}. Also let 
H_\gamma = \{\eta \in \lambda: \gamma \in V(\{\eta\})\} and define S^\delta_\gamma = \{a_\eta(\gamma): \eta \in H_\gamma \cap \textbf{I}_\delta 
and a_\eta(\gamma) \land c_\delta(\gamma) \neq 0\} (and S^\delta_\gamma = \emptyset if this is empty) for 
each \gamma < \alpha and \delta < 2^\alpha. Now if H \in [\textbf{I}_\delta]^\omega, \{\gamma \in \alpha: 
|S^\delta_\gamma| > |H|\} \supset V(H) \cap \{\gamma \in \alpha: c_\delta(\gamma) \land a_\eta(\gamma) \neq 0 for 
\gamma \in H\} \in p. Therefore, by 1.2, |\Pi_{\gamma < \alpha} S^\delta_\gamma/p| = 2^\alpha for each 
\delta \in 2^\alpha. It follows, therefore, that for \delta \in 2^\alpha, we may 
choose d_\delta \in \Pi_{\gamma < \alpha} S^\delta_\gamma/p so that E(p, 0, d_\delta \land a_\eta) for \eta < \lambda and 
\lnot E(p, d_\delta, d_\beta) for \beta < \delta < 2^\alpha. Now let \beta < \delta < 2^\alpha, we show 
that E(p, 0, d_\delta \land d_\beta). Indeed, let \eta_0 \in \textbf{I}_\beta and \eta_1 \in \textbf{I}_\delta be
arbitrary and let \( \gamma \in V(\{\eta_0\}) \cap V(\{\eta_1\}) \cap \{ \gamma \in \alpha : d_\beta(\gamma) \neq d_\delta(\gamma) \} \in p \). Now, by choice of \( \gamma \), if \( d_\delta(\gamma) = a_\eta(\gamma) \) and \( d_\beta(\gamma) = a_\xi(\gamma) \) then \( \{\eta, \xi\} \in H_\gamma \) and so \( \gamma \in V(\{\eta, \xi\}) \subseteq W(\{\eta, \xi\}) \) which implies \( a_\eta(\gamma) \land a_\xi(\gamma) = 0 \). Therefore, for \( \delta < 2^\alpha \) and \( \gamma < \alpha \), let \( e_\delta(\gamma) = d_\delta(\gamma) \land c_\delta(\gamma) \) and we have our disjoint refinement.

Similarly one can prove that if \( \{a_\eta : \eta < \lambda\} \subseteq B^a/p \) is an increasing chain (with \( \lambda \) a limit) then \( C = \{b \in B^a/p : \{\eta : b \land a_\eta = a_\xi \neq 0 \text{ for } \xi < \eta\} \text{ is cofinal in } \lambda\} \) has a disjoint refinement.

3.2 (2) Proof. Let \( x \in S(B^a/p) \) and suppose that \( \{a_\eta : \eta < \lambda\} \subseteq B^a/p \) is chosen with \( \lambda \) minimal such that

\( \{a_\eta : \eta < \lambda\} \) is an increasing chain, \( x \not\in \{a_\eta^* : \eta < \lambda\} \) (i.e. \( a_\eta \not\in x \) for \( \eta < \lambda \)) and for \( a \in x \) there is an \( \eta < \lambda \) with \( a \land a_\eta \neq 0 \) (i.e. \( x \in cl \cup a_\eta^* \)). Let \( a_\lambda = 1 \) and for each \( \gamma \leq \lambda \) with \( cf(\gamma) = \omega \) let \( C_\gamma = \{b \in B^a/p : b \leq a_\gamma \) and \( \{\eta \leq \gamma : b \land a_\eta - a_\xi \neq 0 \text{ for } \xi < \eta\} \) is cofinal in \( \gamma \} \). By Lemma 3.3 (with \( \lambda = \omega \)), the set \( C_\gamma \) has a disjoint refinement \( C'_\gamma \) so that for \( c \in C'_\gamma, c \leq a_\gamma \land a_\eta \) for \( \eta < \gamma \). Therefore

\( \cup\{C'_\gamma : \gamma \leq \lambda \text{ with } cf(\gamma) = \omega\} \) is a disjoint refinement of

\( \cup\{C_\gamma : \gamma \leq \lambda, cf(\gamma) = \omega\} \). To complete the proof it suffices to show that for \( a \in x \) there is a \( \gamma \leq \lambda \) with \( cf(\gamma) = \omega \) and \( a \land a_\gamma \in C_\gamma \). Indeed choose \( \gamma_0 < \lambda \) so that \( a \land a_{\gamma_0} \neq 0 \).

If we have chosen \( \gamma_n < \lambda \) choose \( \gamma_{n+1} < \lambda \) so that \( a - a_{\gamma_n} \land a_{\gamma_{n+1}} \neq 0 \). Now if \( \gamma = sup\{\gamma_n : n \in \omega\} \) we have that \( a \land a_\gamma \in C_\gamma \).
3.2 (3) Proof. This is just 1.4.

3.4 Corollary. $2^\omega > \omega_1$ implies there are $p, q \in U(\omega)$ so that $[\co(2^\omega)]^\omega/p \neq [\co(2^\omega)]^\omega/q$ and $S([\co(2^\omega)]^\omega/p)$ does not map onto $U(\omega)$ by an open map.

Proof. This follows from 2.2, 3.1(3) and 3.2(3).

Let $B = \co(2^\omega)$ and let $M$ be the model of set theory described in 2.9. Kunen has shown that in this model $P(\omega)/\text{fin}$ has no chains of order type $\omega_2$. However if we let $p \in U(\omega)$ be chosen so that $B^\omega/p$ is $\omega_2$-saturated as in 2.9 we have the following result.

3.5 Proposition. It is consistent that there is a $p \in U(\omega)$ such that $P(\omega)/\text{fin}$ embeds into $B^\omega/p$ but $B^\omega/p$ does not embed into $P(\omega)/\text{fin}$. Equivalently $S(B^\omega/p)$ maps onto $U(\omega)$ but $U(\omega)$ does not map onto $S(B^\omega/p)$.

In [BFM], the authors introduce a condition which they call (*) where (*) is the statement "each closed subset of $U(\omega)$ is homeomorphic to a nowhere dense $P_\omega$-set of $U(\omega)$." They show that $\text{CH}$ implies (*) and that $\text{MA + c} = \omega_3$ implies (*) is false. We verify their conjecture that (*) implies $\text{CH}$. A subset of $K$ of a space $X$ is a $P_\alpha$-set if the filter of neighborhoods of $K$ is $\alpha$-complete ($K$ is a $P$-set if it is a $P_\omega$-set).

3.6 Lemma. If $K \subset U(\omega)$ is a closed $P_\alpha$-set and for some $\kappa, \lambda$ with $\omega \leq \kappa \leq \alpha$ and $\omega \leq \lambda$, $\co(K)$ has a $(\kappa, \lambda)$-gap then $\co(U(\omega))$ has a $(\kappa, \lambda')$-gap for some $\omega \leq \lambda' \leq \lambda$. 
Proof. Let \( \{a_\gamma : \gamma < \kappa \} \cup \{b_\beta : \beta < \lambda \} \subset \text{CO}(K) \) so that 
\[ \gamma_1 < \gamma_2 < \kappa \text{ and } \beta_1 < \beta_2 < \lambda \implies a_{\gamma_1} < a_{\gamma_2} < b_{\beta_2} < b_{\beta_1}. \]
Choose \( \{a'_\gamma : \gamma < \kappa \} \subset \text{CO}(U(\omega)) \) so that \( a'_\gamma \cap K = a_\gamma \) for \( \gamma < \kappa \).
For each \( \gamma < \kappa \), we can find \( U_\gamma \in \text{CO}(U(\omega)) \) so that \( K \subset U_\gamma \) 
and \( U_\gamma \cap a'_\gamma - a'_\delta = \emptyset \) for \( \delta < \gamma \). Also since \( \kappa < \omega \), there is 
a \( U \) in \( \text{CO}(U(\omega)) \) with \( K \subset U \) so that \( U \subset U_\gamma \) for \( \gamma < \kappa \). Therefore we may suppose that \( a'_\delta \subset a'_\gamma \) for \( \delta < \gamma < \kappa \). Now, choose 
for as long as possible, \( b'_\beta \in \text{CO}(U(\omega)) \) so that \( b'_\beta \cap K = b_\beta \) 
and \( a'_\gamma \subset b'_\beta \subset b'_\delta \) for \( \delta < \beta \) and \( \gamma < \kappa \). Therefore, for some \( \lambda' < \lambda \), we cannot choose \( b_{\lambda'} \), and we have a gap in \( \text{CO}(U(\omega)) \).

3.7 Proposition. If \( \beta \omega \) embeds into \( U(\omega) \) as a \( P_\alpha \)-set then \( b \geq \alpha \).

Proof. Suppose that \( \{p_n : n \in \omega \} \) is a discrete subset 
of \( U(\omega) \) such that \( K = \text{cl}_{\beta \omega} \{p_n : n \in \omega \} \) is a \( P_\alpha \)-set (it is 
well known that \( K \) is homeomorphic to \( \beta \omega \)). Choose pairwise 
disjoint subsets \( \{A_n : n \in \omega \} \) of \( \omega \) so that \( A_n \in P_n \), and fix 
an indexing \( A_n = \{a(n,m) : m \in \omega \} \) for each \( n \in \omega \). Let 
\( F \subset \omega^\omega \) with \( |F| < \alpha \); we show that \( F \) is bounded. For each 
\( f \in F \), let \( B_f = \{a(n,m) : n \in \omega \text{ and } m > f(n) \} \). Clearly 
\( K \subset B_f^\star \) for \( f \in F \) and so we may choose \( B \subset \omega \) so that \( K \subset B^\star \) 
and \( |B \setminus B_f| < \omega \) for \( f \in F \). Let \( g \in \omega^\omega \) be defined by 
g(n) = \text{min}\{m : a(n,m) \in B \} \text{ and observe that } f <^* g \text{ for } f \in F. 

3.8 Theorem. \( (*) \) is equivalent to \( \text{CH} \).

Proof. Clearly if \( (*) \) is true then \( \beta \omega \) must embed in 
\( U(\omega) \) as a \( P_\omega \)-set. Therefore by 3.7 it suffices to show that 
\( b = \omega_1 \). Now let \( p \in U(\omega) \) be chosen so that \( \kappa(p) = \omega_1 \). 
Let \( \{a_n : n \in \omega \} \subset \text{CO}(U(\omega)) \) be pairwise disjoint and let
\[ \mathcal{K}^P = \cap \left\{ \text{cl}_{\text{U}(\omega)} \ U \left\{ a_n : n \in \omega \right\} : A \in p \right\}. \] It is well known that \( \text{cl}_{\text{U}(\omega)} \ U \left\{ a_n : n \in \omega \right\} = \beta(\omega \times \text{U}(\omega)) \). Therefore \( \text{CO}(\mathcal{K}^P) = [\text{CO}(\text{U}(\omega))]^\omega/p \) and by 1.4 has an \((\omega, \omega_1)\)-gap. By 3.6, if \( \mathcal{K}^P \) embeds into \( \text{U}(\omega) \) as a \( \mathcal{P}_c \)-set with \( \mathcal{C} > \omega_1 \) then \( b = \omega_1 \).

3.9 Remark. It is not difficult to show that if \( A \) is a boolean algebra which has an \((\omega_1, \omega_1)\)-gap then so does \( A^\omega/p \) for each \( p \in \text{U}(\omega) \) and is therefore not \( \omega_2 \)-saturated. This means that we cannot easily obtain compact subsets \( K \) of \( \text{U}(\omega) \) so that \( \text{CO}(K) \) is \( \omega_2 \)-saturated (such as subsets of the boundary of a cozero set). However \( S(B^\omega/p) = \mathcal{K}^P \) as in 3.5 is in some sense a "well-placed" subset of \( \beta(\omega \times 2^\omega) \). For instance \( \mathcal{K}^P \) is a \( 2^\omega \)-set in \( (\omega \times 2^\omega)^* = \beta(\omega \times 2^\omega) \setminus (\omega \times 2^\omega) \) (see [BV]). Furthermore we can easily construct \( p \) to be \( 2^\omega \)-OK (see [K2]) in which case every \( \text{ccc} \) subspace of \( (\omega \times 2^\omega)^* \) meets \( \mathcal{K}^P \) in a nowhere dense set. Furthermore if we use 2.10 to find \( p \) a \( \mathcal{P}_c \)-point then \( \mathcal{K}^P \) is a \( \mathcal{P}_c \)-set in \( (\omega \times 2^\omega)^* \). I do not know if it is possible to find a \( \mathcal{P}_{\omega_2} \)-set \( K \) in \( \text{U}(\omega) \) such that \( \text{CO}(K) \) is \( \omega_2 \)-saturated. Although Shelah has found a model in which \( \text{U}(\omega) \) is not homeomorphic to \( (\omega \times 2^\omega)^* \) (see [vM]) it would be interesting if they were not in one of the above models.

After acceptance of this paper, John Merrill brought it to the author's attention that 2.2 and a more general version of 2.3 appear in Shelah's Model Theory book. However as the proofs presented here seem simpler we have chosen to include them.
References


[vD] E. K. van Douwen, Transfer of information about \( \mathbb{N} \) via open remainder maps (preprint).


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