ON NON-METRIC PSEUDO-ARCS

by

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We construct an example of non-metric hereditarily indecomposable continuum that has many of the properties of the pseudo-arc. In particular, we construct a non-metric hereditarily indecomposable homogeneous hereditarily equivalent continuum.

Definitions. A continuum is defined to be a compact connected Hausdorff space. Suppose $\lambda$ is an ordinal, $X_\alpha$ is a topological space for each $\alpha < \lambda$, and if $\alpha < \beta$ then $h^\beta_\alpha$ is a mapping from $X_\beta$ to $X_\alpha$. Then the space $X = \lim_{\alpha < \beta < \lambda} \{X_\alpha, h^\beta_\alpha\}$ denotes the space which is the inverse limit of the inverse system $\{X_\alpha, h^\beta_\alpha\}_{\alpha < \beta < \lambda}$. Each point of $X$ is a function $P: \lambda \to \bigcup_{\alpha < \lambda} X_\alpha$ such that for all $\alpha < \beta < \lambda$: $P(\alpha) = P_\alpha \in X_\alpha$ and $P_\alpha = h^\beta_\alpha(P_\beta)$. A basis for the topology is the collection to which the set $U$ belongs if and only if there exists a $\beta < \lambda$ and an open set $O_\beta$ of $X_\beta$ so that $U = \{P | P_\beta \in O_\beta\}$.

Let $\pi_\alpha: X \to X_\alpha$ be defined by $\pi_\alpha(P) = P_\alpha$.

Suppose that $M$ is a continuum and $P \in M$. Then $C$ is the composant of $M$ at $P$ means that $C$ is the point set to which $x$ belongs if and only if there is a proper subcontinuum of $M$ containing $x$ and $P$. The set $C$ is a composant of $M$ means that $C$ is a composant of $M$ at some point of $M$. A pseudo-arc is a nondegenerate hereditarily indecomposable metric chainable continuum. The pseudo-arc is homogeneous [Bi] and hereditarily equivalent [Ms].
We use the following results due to Wayne Lewis [L2].

**Theorem A.** Suppose that $M$ is a one-dimensional continuum. Then there exists a one dimensional continuum $\hat{M}$ and a continuous decomposition $G$ of $\hat{M}$ into pseudo-arcs so that the decomposition space $\hat{M}/G$ is homeomorphic to $M$. Furthermore, if $\pi: \hat{M} \rightarrow \hat{M}/G$ is the mapping so that $\pi(x)$ is the element of $G$ containing $x$ then if $h: \hat{M}/G \rightarrow \hat{M}/G$ is a homeomorphism then there exists a homeomorphism $\hat{h}: \hat{M} \rightarrow \hat{M}$ so that $\pi \circ \hat{h} = h \circ \pi$.

**Theorem B.** Under the hypothesis of theorem A if $x$ and $y$ are elements of the same pseudo-arc in $G$ then there exists a homeomorphism $\hat{h}: \hat{M} \rightarrow \hat{M}$ so that $\hat{h}(x) = y$ and $\pi \circ \hat{h} = \pi$.

From the fact that the pseudo-arc of pseudo-arcs is unique [L1], we have the following:

**Corollary B.** Suppose that $X$ is a pseudo-arc and $G$ is a continuous collection of pseudo-arcs filling $X$, so that for each $x \in X$, $\pi(x)$ is the element of $G$ that contains $x$, and $Y = X/G$. Then $Y$ is a pseudo-arc and if $h: Y \rightarrow Y$ is a homeomorphism then there exists a homeomorphism $\hat{h}: X \rightarrow X$ so that $\pi \circ \hat{h} = h \circ \pi$.

**Example 1.** Let $X_1$ be a pseudo-arc, let $X_2$ be a pseudo-arc, and let $G_2$ be a continuous decomposition of $X_2$ into pseudo-arcs. Then $X_2/G_2$ is a pseudo-arc and is homeomorphic to $X_1$. Let $f_2^1$ be the open monotone map, $f_2^2: X_2 \rightarrow X_1$ so that $G_2 = \{f_2^{2-1}(x) | x \in X_1\}$. By induction, construct $\{X_\alpha\}_{\alpha < \omega_1}$ as follows. Suppose $\gamma < \omega_1$ and $X_\alpha$ and $f_\alpha^\beta$ have been
constructed for all $\alpha$ and $\beta$ such that if $\alpha < \beta < \lambda$ then $X_\alpha$ is a pseudo-arc and $f_\alpha^\beta: X_\beta \to X_\alpha$ is an open monotone map.

Suppose $\lambda < \omega_1$ is not a limit ordinal. Then $\lambda$ has a predecessor $\lambda - 1$. Then let $X_\lambda$ be a pseudo arc and let $G_\lambda$ be a continuous decomposition of $X_\lambda$ into pseudo-arcs. Then $X_\lambda/G_\lambda$ is homeomorphic to $X_{\lambda-1}$ so there is an open monotone map $f_\lambda^\lambda_{\lambda-1}: X_\lambda \times X_{\lambda-1}$ so that $G_\lambda = \{ f_\lambda^\lambda_{\lambda-1}(x) \mid x \in X_{\lambda-1} \}$. For $\alpha < \lambda - 1$ let $f_\alpha^\lambda = f_\lambda^\lambda_{\lambda-1} \circ f_\alpha^{\lambda-1}$. Suppose that $\lambda$ is a limit ordinal. Then $\{ (X_\alpha, f_\alpha^\beta) \}_{\alpha < \beta < \lambda}$ is an inverse system. Let $X_\lambda = \lim_{\alpha < \beta < \lambda} (X_\alpha, f_\alpha^\beta)$. If $\lambda < \omega_1$ then some countable set is cofinal in $\lambda$ so $X_\lambda$ is homeomorphic to an inverse limit of pseudo-arcs and hence must be a metric chainable hereditarily indecomposable continuum. So $X_\lambda$ is a pseudo-arc. If $\alpha < \lambda$ then let $f_\alpha^\lambda: X_\lambda \to X_\alpha$ denote the projection of $X_\lambda$ onto the $\alpha$-th coordinate space $X_\alpha$.

Let $M$ denote the space $X = \lim_{\alpha < \beta < \omega_1} (X_\alpha, f_\alpha^\beta)$.

**Theorem 1.1.** The space $M$ is a non-metric chainable hereditarily indecomposable continuum.

**Proof.** The chainability and hereditary indecomposability of $M$ easily follows from the fact that each $X_\alpha$ is chainable and hereditarily indecomposable. The non-metrizability of $M$ follows from the existence of an $\omega_1$-long monotonic sequence of subcontinua of $M$ which is constructed below.

Let $L_1 = X_1$, $I_1 = X_{\omega_1}$, and $P_1 \in L_1$. Let $I_2 = \{ x \in M \mid x_1 = P_1 \}$, $L_2 = \{ x_2 \in X_2 \mid x \in I_2 \} = \pi_2(I_2)$, and $P_2 \in L_2$. By
the construction of $X_\alpha$, $L_2$ is nondegenerate and in fact

$$L_2 = f_1^{2-1}(P_1).$$

Let $\lambda < \omega_1$.

Suppose $I_\alpha$, $P_\alpha$, and $L_\alpha$ have been constructed for all $\alpha < \lambda$.

Case i: $\lambda$ is not a limit ordinal and $\lambda = \lambda' + 1$. Then let $I_\lambda = \{x| x_\lambda = P_\lambda, \}, L_\lambda = \{x_\lambda \in X_\lambda | x \in I_\lambda \} = \pi_\lambda(I_\lambda)$, and $P_\lambda \in L_\lambda$.

Case ii: $\lambda$ is a limit ordinal. Then let $I_\lambda = \cap_{\alpha < \lambda} I_\alpha$, $L_\lambda = \pi_\lambda(I_\lambda)$, and $P_\lambda \in L_\lambda$.

Note that if $\alpha \neq \beta$ then $I_\alpha \neq I_\beta$ and if $\alpha < \beta$ then $I_\beta \subseteq I_\alpha$. So $\{I_\lambda| \lambda < \omega_1 \}$ is the required monotonic collection.

**Theorem 1.2.** The space $M$ is homogeneous.

**Proof.** Let $x$ and $y$ be two points of $M$. Since $X_1$ is homogeneous there exists a homeomorphism $h: X_1 \rightarrow X_1$ so that $h(x_1) = y_1$. By theorem A there is a homeomorphism $g: X_2 \rightarrow X_2$ so that $h \circ f_2^1 = f_2^1 \circ g$. Note that $f_2^1 \circ g(x_2) = h \circ f_1^2(x_2) = h(x_1) = y_1$ and $f_2^2(y_2) = y_1$. So $g(x_2)$ and $y_2$ both belong to the same element of $G_2$. So by theorem B there exists a homeomorphism $k: X_2 \rightarrow X_2$ so that $k \circ g(x_2) = y_2$ and $f_1^2 \circ k = f_1^2$. Thus $k \circ g: X_2 \rightarrow X_2$ is a homeomorphism with $f_1^2 \circ k \circ g = f_1^2 \circ g = h \circ f_1^2$ and $k \circ g(x_2) = y_2$. Define $\theta_1 = h$, and $\theta_2 = k \circ g$. Thus $\theta_1 \circ f_1^2 = f_1^2 \circ \theta_2$.

Proceeding by induction, suppose that $\lambda < \omega_1$ and $\theta_\alpha$ has been defined for all $\alpha < \lambda$ so that if $\alpha < \beta < \lambda$ then $\theta_\alpha \circ f_\alpha^\beta = f_\alpha^\beta \circ \theta_\beta$.

Case i: $\lambda$ is not a limit ordinal and $\lambda = \lambda' + 1$ for some $\lambda'$. Then using the same argument as above there exists
\[ \theta_\lambda : X_\lambda \to X_\lambda \text{ so that } \theta_\lambda \circ f_\lambda^\alpha = f_\lambda^\alpha \circ \theta_\lambda \text{ and } \theta_\lambda(x_\lambda) = y_\lambda. \]

If \( \alpha < \lambda \), then \( \theta_\alpha \circ f_\lambda^\alpha = \theta_\alpha \circ f_\alpha^\lambda = f_\lambda^\alpha \circ \theta_\lambda \circ f_\lambda^\alpha = f_\alpha^\lambda \circ f_\lambda^\alpha \circ \theta_\lambda = f_\alpha^\lambda \circ \theta_\lambda. \]

Case ii: \( \lambda \) is a limit ordinal. Then, since \( X_\lambda \) is the inverse limit \( \lim_{\alpha < \beta < \lambda} \{X_\alpha, f_\alpha^\beta\} \), the collection \( \{\theta_\lambda : X_\lambda \to X_\lambda; \alpha < \lambda\} \) induces a homeomorphism \( \theta_\lambda : X_\lambda \to X_\lambda \) so that \( \theta_\lambda \circ f_\alpha^\lambda = f_\lambda^\alpha \circ \theta_\lambda. \)

Then since \( M = X_{\omega_1} \) is the inverse limit \( \lim_{\alpha < \beta < \omega_1} \{X_\alpha, f_\alpha^\beta\}. \)

The collection \( \{\theta_\alpha\}_{\alpha < \lambda} \) induces a homeomorphism \( \theta : X_{\omega_1} \to X_{\omega_1} \) so that \( f_{\omega_1} \circ \theta = \theta_\alpha \circ f_\alpha^\omega \) and since \( \theta_\lambda(x_\lambda) = y_\lambda \) we also have \( \theta(x) = y. \)

Definition. The continuum \( X \) is said to be hereditarily equivalent if it is homeomorphic to each of its nondegenerate subcontinua.

Theorem 1.3. The space \( M \) is hereditarily equivalent.

Proof. Let \( L \) be a nondegenerate subcontinuum of \( M. \)

Let \( P \) and \( Q \) be two points of \( L. \) Then there exists \( \lambda < \omega_1 \) so that \( P_\lambda \neq Q_\lambda. \) Let \( L_\alpha \) denote the projection of \( L \) into the \( \alpha^{th} \) coordinate. Thus \( L_\alpha = \{x_\alpha | x \in L\} = f_\alpha^{\omega_1}(L). \) First we will show that if \( \lambda < \gamma < \omega_1 \) then

\[ L_\gamma = f_\lambda^{\gamma-1}(L_\lambda). \]

Clearly, \( L_\gamma \subseteq f_\lambda^{\gamma-1}(L_\lambda). \)

For each \( x \in L_\lambda \) the set \( f_\lambda^{\gamma-1}(x) \) is a subcontinuum of \( X_\lambda. \) Since \( L_\lambda \) is nondegenerate, it follows that \( L_\gamma \) is not a subset of \( f_\lambda^{\gamma-1}(x). \) But by hereditary indecomposability one
of $L_{\gamma}$ and $f_{\lambda}^{-1}(x)$ is a subset of the other. So $f_{\lambda}^{-1}(L_{\lambda}) \subset L_{\lambda}$. Therefore we have $L_{\gamma} = f_{\lambda}^{-1}(L_{\lambda})$. Notice that this argument also verifies that $f_{\lambda}|L_{\gamma} : L_{\gamma} + L_{\lambda}$ is a monotone map. Thus $L = \lim_{\lambda < \alpha < \beta \in \omega_1} \{L_{\alpha}, f_{\beta}|L_{\beta}\}$.

The set $\omega_1$ is order isomorphic to the set $\{\gamma | \lambda < \gamma < \omega_1\}$. Let $\psi$ be the isomorphism. Suppose $\lambda < \omega_1$ and $\{\theta_{\alpha}\}_{\alpha < \lambda}$ have been defined so that for all $\alpha < \beta < \lambda$

$$\theta_{\alpha} \circ f_{\alpha} = f_{\psi}(\beta) | L_{\psi}(\beta) \circ \theta_{\beta}.$$  

If $\lambda$ is not a limit ordinal and $\lambda = \gamma + 1$ then using Wayne Lewis's results there exists a homeomorphism $\theta_{\gamma+1} : X_{\gamma+1} + L_{\psi}(\gamma+1)$ so that the following diagram commutes

$$
\begin{array}{ccc}
X_{\gamma} & \overset{f_{\gamma+1}}{\rightarrow} & X_{\gamma+1} + \cdots + X_{\beta} \\
\theta_{\gamma} \downarrow & & \theta_{\gamma+1} \downarrow \\
L_{\psi}(\gamma) & \leftarrow & L_{\psi}(\gamma+1) + \cdots + L_{\psi}(\beta) \\
\end{array}
$$

If $\lambda$ is a limit ordinal the maps $\{\theta_{\gamma}\}_{\gamma < \lambda}$ induce a homeomorphism $\theta_{\lambda}$ of $X_{\lambda}$ onto $X_{\psi}(\lambda)$. Therefore for all $\alpha < \beta < \omega_1$

$$\theta_{\alpha} \circ f_{\alpha} = f_{\psi}(\beta) | L_{\psi}(\beta) \circ \theta_{\beta}$$

and the maps $\{\theta_{\gamma}\}_{\gamma < \omega_1}$ induce a homeomorphism of $M$ onto $L$.

**Theorem 1.4.** The continuum $M$ is irreducible from the point $x_1$ to the point $y_1$ if and only if $X_1$ is irreducible from the point $x_1$ to the point $y_1$.
Proof. Suppose that \( X_1 \) is not irreducible from \( x_1 \) to \( y_1 \). Then there is a proper subcontinuum \( L_1 \) of \( X_1 \) containing \( x_1 \) and \( y_1 \). Let \( L_2 = \frac{f_{X_1}^{2-1}}{X_1} (L_1) \); then, since \( f_{X_1}^2 \) is monotone, \( L_2 \) is a subcontinuum of \( X_2 \) and it must be a proper subcontinuum of \( X_2 \) because \( L_1 \) is proper in \( X_1 \). Let us construct a collection \( \{ L_\alpha \}_{\alpha < \omega_1} \) by induction so that \( L_\alpha \) is a proper subcontinuum of \( X_\alpha \) containing \( x_\alpha \) and \( y_\alpha \). Suppose that \( L_\alpha \) has been defined for all \( \alpha \in \lambda \). If \( \lambda \) is not a limit ordinal then let \( L_\lambda = f_{X_\lambda}^{\lambda-1} (L_{\lambda-1}) \). Since \( f_{X_\lambda}^{\lambda} \) is monotone and \( L_{\lambda-1} \) is a proper subcontinuum of \( X_{\lambda-1} \) then \( L_\lambda \) is a proper subcontinuum of \( X_\lambda \), and \( L_\lambda \) contains \( x_\lambda \) and \( y_\lambda \). If \( \lambda \) is a limit ordinal then let \( L_\lambda = \lim \{ X_\alpha, f_{X_\alpha}^{\beta} | L_\beta \} \). Since \( L_1 \) is a proper subcontinuum of \( X_1 \) then \( L_\lambda \) is a proper subcontinuum of \( X_\lambda \).

Since \( x_\alpha \) and \( y_\alpha \) lie in \( L_\alpha \) for \( \alpha < \lambda \), and for \( \alpha < \beta < \lambda \) \( f_{\alpha}^{\beta}(x_\alpha) = x_\alpha \) and \( f_{\alpha}^{\beta}(y_\alpha) = y_\alpha \), then \( x_\lambda \) and \( y_\lambda \) lie in \( L_\lambda \).

Therefore \( L = \lim \{ L_\alpha, f_{\alpha}^{\beta} | L_\beta \} \) is a proper subcontinuum of \( M \). Furthermore by construction for each \( \alpha < \omega_1 \) the points \( X_\alpha \) and \( y_\alpha \) both lie in \( L_\alpha \). So \( L \) contains \( x \) and \( y \) hence \( M \) is not irreducible from \( x \) to \( y \).

Suppose that \( M \) is not irreducible from the point \( x \) to the point \( y \). Let \( L \) be a proper subcontinuum of \( M \) containing \( x \) and \( y \). Then for some \( \lambda < \omega_1 \), \( f_{X_\lambda}^{\omega_1}(L) \neq X_\lambda \). Let \( L_\lambda = f_{X_\lambda}^{\omega_1}(L) \). Then \( x_\lambda \) and \( y_\lambda \) both lie in \( L_\lambda \). Since \( L_\lambda \) is a proper subcontinuum of \( X_\lambda \) there is a point \( z \in X_\lambda - L_\lambda \).

Let \( z_1 = f_{X_1}^{\lambda-1}(z_\lambda) \). Then \( f_{X_1}^{\lambda-1}(z_1) \) is a subcontinuum of \( X_\lambda \).

But \( z_\lambda \in f_{X_1}^{\lambda-1}(z_1) \) and \( z_\lambda \not\in L_\lambda \) also \( x_\lambda \in L_\lambda \) so \( x_1 \neq z_1 \) and
hence $x_1 \not\in f_1^{-1}(z_1)$. So by hereditary indecomposability, $L_\lambda$ and $f_1^{-1}(z_1)$ are disjoint continua. Thus $z_1 \not\in f_1^\lambda(L_\lambda)$ but $x_1$ and $y_1$ are elements of $f_1^\lambda(L_\lambda)$. Therefore, $f_1^\lambda(L_\lambda)$ is a proper subcontinuum of $X_1$ containing $x_1$ and $y_1$.

The following corollary follows easily from the construction and theorem 1.4.

Corollary 1.5. The continuum $M$ has $c$ composants.

Example 2. In [S3] an example of a hereditarily indecomposable continuum with exactly two composants was constructed. The example was an inverse limit of pseudo-arcs indexed by $\omega_1$ with special types of retractions as bonding maps.

We will use the following theorems from [S3].

Theorem C. Suppose that $X$ is a pseudo-arc, $X$ is irreducible from the point $P$ to the point $Q$, $Y$ is a pseudo-arc, $X \subset Y$, and $Y$ is the union of two closed sets $H$ and $K$ so that $X$ is a component of $H$, $X \cap K = \{Q\}$, and $\text{Bd}(H) = \text{Bd}(K) = K \cap H$. Then there is a retraction $h$ of $Y$ onto $X$ so that $h(K) = Q$, $h^{-1}(P) = P$, and $h(Y-X)$ lies in the composant of $X$ at $Q$.

Suppose $X$ is a continuum. Let us use the following notation. If $H \subset X$, let $\text{Bd}_X(H)$ denote the boundary of $H$ in $X$, let $\text{Int}_X(H)$ denote the interior of $H$ with respect to $X$, and let $\text{Cl}_X(H)$ denote the closure of $H$ in $X$. If $Q \in X$, then let $\text{Cmps}(X,Q)$ denote the composant of $X$ at $Q$. 

Theorem C was used to construct the example in [S3].

The example which we will denote by \(N\) was constructed so that

\[
N = \lim_{\alpha < \omega_1} \{ X_\alpha, h_\alpha^\beta \}
\]

and for each \(\alpha < \omega_1:\)

1) \(X_\alpha\) is a pseudo-arc with \(X_\alpha \subseteq X_{\alpha+1}\),
2) \(X_\alpha\) is irreducible from the point \(P\) to the point \(Q_\alpha\),
3) \(X_{\alpha+1}\) is the union of two closed sets \(H_{\alpha+1}\) and \(K_{\alpha+1}\) so that \(X_\alpha\) is a component of \(H_{\alpha+1}\), \(X_{\alpha+1} \cap K_{\alpha+1} = \{Q_\alpha\}\),
4) \(h_{\alpha+1} : X_{\alpha+1} \to X_\alpha\) is a retraction so that \(h_{\alpha+1}(K_{\alpha+1}) = Q_\alpha\), \(h_{\alpha+1}(P) = P\), and \(h_{\alpha+1}(X_{\alpha+1} - X_\alpha) \subseteq \text{Cmps}(X_\alpha, Q_\alpha)\).

Conditions 1-4 were used to obtain the following theorem [S].

**Theorem D.** The continuum \(N = \lim_{\alpha < \omega_1} \{ X_\alpha, h_\alpha^\beta \}\) is a hereditarily indecomposable continuum with exactly two composants.

By Theorem D it follows that \(N\) is a non-metric continuum. By Theorem D and Corollary 1.5 the continua \(M\) and \(N\) are not homeomorphic. It would be of interest to determine if \(N\) is homogeneous or hereditarily equivalent. We will show that \(N\) is neither of these, and we will obtain a general theorem about non-metric hereditarily indecomposable continua.

The fact that \(N\) is not hereditarily equivalent easily follows from the following observation.
Theorem 2.1. The continuum $\mathbb{N}$ contains a pseudo-arc.

Proof. The proof easily follows from the construction.

From condition 4 $h_{a+1}^a: X_{a+1} \rightarrow X_a$ is a retraction and $h_{a+1}^a(X_{a+1} - X_a) \subset \text{Cmps}(X_a, Q_a)$. So if $I$ is a proper subcontinuum of $X_a$ that does not intersect $\text{Cmps}(X_a, Q_a)$ then $f_{a+1}^a(I) = I$. Therefore, if $L$ is a nondegenerate subcontinuum of $X_1$ that does not intersect $\text{Cmps}(X_1, Q_1)$, then $f_{1}^{-1}(L) = L$. So $\hat{L} = \lim_{a < \beta < \omega_1} \{L, f_{a}^{\beta} \mid L\}$ is a pseudo-arc since $f_{a}^{\beta} \mid L$ is the identity on $L$.

Definitions. Suppose $X$ is a space and $x \in X$. Then $X$ is first countable at $x$ means that there is a countable collection of open sets that forms a basis at $x$. The point $x$ is a P-point of $X$ means that if $\{O_i\}_{i=1}^\infty$ is a countable collection of open sets each containing $x$, then there exists an open set $O$ containing $x$ such that $O \subset \bigcap_{i=1}^\infty O_i$.

The fact that $\mathbb{N}$ is not homogeneous easily follows from Theorems D and 2.1 as well as from the following theorem.

Theorem 2.2. The continuum $\mathbb{N}$ contains both a point at which it is first countable and a P-point.

Proof. First we show that the point $Q = \{Q_a\}$ is a P-point of $\mathbb{N}$. Suppose $\alpha < \omega_1$ and $R$ is an open set in $X_{\alpha}$ then let $\hat{R} = \{x \in N \mid x_{\alpha} \in R\}$, the set $\hat{R}$ is open in $N$. Suppose $\{O_i\}_{i=1}^\infty$ is a countable sequence of open sets in $N$ each containing $Q$. Then for each $i$ there is an ordinal $\alpha_i$ and an open set $R_i$ in $X_{\alpha_i}$, so that $Q_{\alpha_i} \in R_i$ and $Q \in \hat{R}_i \subset O_i$. 

Since \( \{a_1\}_{i=1}^{\infty} \) is countable there exists \( \lambda < \omega_1 \) so that 
\( a_i < \lambda \) for all positive integers \( i \) and so that \( \lambda \) is not a 
limit ordinal. Let \( U \) be an open set containing \( Q_\lambda \) so that 
\( \text{Cl}_{x_1}(U) \subset K_\lambda \), this can be done by condition 3. Then by 
condition 4, \( f_{\lambda-1}^{\lambda}(\text{Cl}_{x_1}U) = Q_{\lambda-1} \) and hence \( \tilde{U} \subset O_{a_1} \) for all 
\( a_1 \).

Now we prove that if \( x \in X_1 - \text{Cmps}(X_1, Q_1) \) then the 
point \( z \in N \) so that \( z_\alpha = x \) for all \( \alpha < \omega_1 \) is a point of 
first countability of \( N \). Let \( \{U_i\}_{i=1}^{\infty} \) be a countable local 
basis of open sets of \( X_1 \) at \( x \). We claim that \( \{U_i\}_{i=1}^{\infty} \) 
is a local basis for \( z \) in \( N \). Suppose on the other 
hand that \( \{U_i\}_{i=1}^{\infty} \) is not a local basis for \( z \). Then 
there is a point \( y \neq z \) so that \( y \in \bigcap_{i=1}^{\infty} U_i \). Since \( y \neq z \) 
there is a first \( \lambda \) so that \( y_\lambda \neq z_\lambda = x \). Clearly \( \lambda \) is not 
a limit ordinal and \( \lambda \neq 1 \) since \( \{U_i\}_{i=1}^{\infty} \) is a local basis 
for \( x \in X_1 \). Therefore, \( f_{\lambda-1}^{\lambda}(y_\lambda) = x \). But \( x \in X_1 \subset X_{\lambda-1} \subset X_{\lambda} \) 
and \( f_{\lambda-1}^{\lambda}(X_{\lambda} - X_{\lambda-1}) \subset \text{Cmps}(X_{\lambda-1}, Q_{\lambda-1}) \). Also, \( x \notin \text{Cmps}(X_{\lambda-1}, Q_{\lambda-1}) \) because for \( \lambda = 2 \) \( x \) was chosen so that \( x \notin \text{Cmps}(X_1, Q_1) \) 
and for \( \lambda > 2 \), \( X_1 \) is a proper subcontinuum of \( X_{\lambda-1} \) that 
contains \( P \) and hence cannot intersect \( \text{Cmps}(X_{\lambda-1}, Q_{\lambda-1}) \). 
Therefore, the only point of \( X_{\lambda} \) that is mapped onto \( x \) by 
\( f_{\lambda-1}^{\lambda} \) is \( x \). But this contradicts the fact that \( y_\lambda \neq x \). 
So \( N \) is first countable at \( x \). Similarly it can be shown 
that if \( \lambda < \omega_1 \) and \( x \in X_\lambda - \text{Cmps}(X_\lambda, Q_\lambda) \) then \( N \) is first 
countable at the point \( z \) so that \( z_\alpha = x \) for all \( \lambda < \alpha < \omega_1 \).

The next theorem shows that, in terms of the existence 
of points of first countability and P-points in hereditarily
indecomposable continua example 2 is as complicated as it can get.

**Theorem 3.** If $X$ is a hereditarily indecomposable continuum then no proper subcontinuum of $X$ can contain a P-point of $X$ and a point at which $X$ is first countable.

**Proof.** Suppose $X$ is a hereditarily indecomposable continuum, $x$ is a P-point of $X$, $y$ is a point of $X$ at which $X$ is first countable, and $L$ is a proper subcontinuum of $X$ containing both $x$ and $y$. Let $\{R_i\}_{i=1}^\infty$ be a countable local basis at $y$ so that $R_{i+1} \subset R_i$. Let $z \in X - L$.

Let $I_n$ be the component of $X - R_n$ containing $z$. Then $I_n \cap \text{Bd}_X(R_n) \neq \emptyset$, and since $y \not\in I_n$ by hereditary indecomposability $I_n \cap L = \emptyset$. Thus $x \not\in I_n$. Let $K$ be the limiting set of $I_1, I_2, \ldots$. Since $x$ is a P-point then $x \not\in K$. Since $y$ is the sequential limit of $\{I_n \cap \text{Bd}_X(R_n)\}_{n=1}^\infty$ and $I_n \subset I_{n+1}$ for each $n$ then $K$ is a continuum that contains $y$. Thus $y \in L$, $y \in K$, $z \in K$, $z \not\in L$, $x \not\in K$, and $x \in L$; but this contradicts the hereditary indecomposability of $X$.

The following questions arise naturally from our discussion.

**Question 1.** Are there other non-metric hereditarily equivalent continua?

**Question 2.** Are there other non-metric homogeneous chainable continua? In particular, is there an inverse limit on a larger index set of chainable continua which is homogeneous?
Question 3. How many different inverse limits of pseudo-arcs indexed by $\omega_1$ are there?

Bibliography


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