ON THE COUNTABLE BOX PRODUCT OF COMPACT ORDINALS

by

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If $X$ is a topological space, then $\sigma^K X$ (the box product of $K$ many copies of $X$) denotes the product $\prod^K X$ with the topology induced by the family of all sets of the form $\prod_{\alpha \in K} U_{\alpha}$, where each $U_{\alpha}$ is an open set in $X$. For a recent survey on box products, see [Wi2].

Consider the following theorem due to M. E. Rudin:

0.1 Theorem. Assume the Continuum Hypothesis holds. Then, for each ordinal $\lambda$, $\sigma^\omega \lambda + 1$ is paracompact.

The conclusion to this theorem has been expanded to the larger class of compact spaces ([Kul]) and $\omega_1$ many factors ([Wi3]). Under the set-theoretic statement—there is $\kappa$-scale in $\omega_\omega$—the best result was "$\sigma^{\omega_1} + 1$ is paracompact" ([Wil]). We offer our main result:

0.2 Theorem. Suppose that for some cardinal $\kappa$ there is a $\kappa$-scale in $\omega_\omega$. Then, for each ordinal $\lambda$, $\sigma^\omega \lambda + 1$ is paracompact.

1. Preliminaries

Given a set $X$, $^{\omega}X$ is the set of functions from $\omega$ to the set $X$. For $f$ and $g$ in $^{\omega}X$, define $f =^* g$ if they differ on

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only finitely many coordinates. We denote the resulting quotient set by $V^\omega X$ and write $[f] = \{g: g =^* f\}$.

Suppose $X$ is an ordinal set. There are two very different but similarly defined orders on $V^\omega X$. First of all, define $f <^* g$ ($f, g \in \omega^\omega X$) provided that $f(n) > g(n)$ for only finitely many $n \in \omega$; define $f <^* g$ provided that $f(n) > g(n)$ for only finitely many $n \in \omega$. Then we define

$$[f] <^* [g] \text{ if } f <^* g;$$

$$[f] <^* [g] \text{ if } f <^* g.$$

It is trivial that $[f] = [g]$ iff $f =^* g$ and both orders, $<^*$ and $<^*$, are partial orders on $V^\omega X$.

In this paper, for each $x \in V^\omega X$, we fix some $f_x \in X$ and identify $x = [f_x]$ with $f_x$.

Suppose, $f, g \in V^\omega X$. We define

$$[f, g] = \{h \in V^\omega X: f <^* h <^* g\} = \bigvee_{n \in \omega} \{f(n), g(n)\},$$

and call $[f, g]$ basic set iff both sets $\{n: g(n) \text{ is limit ordinal}, f(n) = g(n)\}$ and $\{n: f(n) \text{ is a limit ordinal}\}$ are finite.

Suppose $\kappa$ is a cardinal. The statement there is a $\kappa$-scale in $\omega^\omega$ means there is an order preserving injection from $\kappa$ into $\omega^\omega$ whose range is confinal in $(\omega^\omega, <^*)$.

Suppose $Z$ is a topological space. Then $V^\omega Z$ denoted the quotient space induced by $=^*$ on $\omega^\omega Z$. This is known as the nabla product. We make strong use of an important lemma due to K. Kunen (see [Wi2]):

1.1 Lemma. If $Z$ is locally compact and paracompact then

(1) $V^\omega Z$ is paracompact iff $\omega^\omega Z$ is paracompact;

(2) $V^\omega Z$ is a P-space (every $G_\delta$-set is open).
So we need only to prove $\forall \omega_\lambda + 1$ is paracompact in order to prove 0.2.

1.2 Definition. A space $X$ is called specially paracompact provided that each open cover of $X$ has a refinement consisting of pairwise disjoint basic sets.

The symbol $\#(a)$ denotes the statement: $\forall \omega_a + 1$ is specially paracompact. According to 1.1 $\#(a)$ implies $\forall \omega_a + 1$ is paracompact.

1.3 Theorem. If there is a $\kappa$-scale in $\omega$, then $\#(\omega_1)$ is true.

This theorem has been proved by Williams in [Wil]. But he stated in [Wil] a weaker proposition: $\forall \omega_\omega + 1$ is paracompact if $\exists$ is a $\kappa$-scale in $\omega$. In fact, his proof really is a stronger one.

1.4 Lemma. Suppose $\lambda$ is an ordinal. Then every clopen set in $\forall \omega_\lambda + 1$ is a union of pairwise disjoint basic sets.

This result is implicitly proved in [Ru]. But it is not so easy to extract from Rudin's paper. Fortunately, in this paper, we need only some particular cases of the lemma: first case, the clopen set is a difference set between two basic sets; second case, the clopen set is an intersection of countably many basic sets. Both are not so hard to prove. We leave it to the readers.
2. Tops Refinement

2.1 Definition. A set \( M \subseteq \omega \times (\lambda + 1) \) is called a matrix provided that there is \( b \in \lambda + 1 \) for each \( n < \omega \) such that \( (n, b) \in M \).

2.2 Definition. Suppose \( f \in V^{\omega \lambda + 1} \). If there is a member \( (n, b) \) of \( M \) for all but finitely many \( n \), such that \( f(n) = b \), then we say that \( f \) is on the matrix \( M \). The set \( \{ f \in V^{\omega \lambda + 1} : f \text{ is on the matrix } M \} \) is denoted by \( D(M) \).

2.3 Lemma. Assume \( \#(\omega_1) \) is true. If a matrix \( M \) is countable and \( D(M) \) is closed, then \( D(M) \) is specially paracompact.

Proof. Since \( M \) is countable and \( D(M) \) is closed, it is easy to find an embedding map \( E \) from a basic set \( [\emptyset, g] \subseteq V^{\omega_1 + 1} \) into \( V^{\omega \lambda + 1} \), such that \( E([\emptyset, g]) = D(M) \), where \( \emptyset = (0, 0, \cdots) \) and \( g < star \omega_1 = (\omega_1, \omega_1, \cdots) \). In fact, \( M_n = \{ b \in \lambda + 1 : (n, b) \in M \cap (\{ n \} \times (\lambda + 1)) \} \) is countable and is a closed set for all but finitely many \( n \).

Let \( n_n \) be the order type of \( M_n \). If \( M_n \) is closed, then \( n_n \) is a successor ordinal, \( n_n = \mu_n + 1 \). Let \( g = (\mu_0, \mu_1, \cdots, \mu_n, \cdots) \). Obviously, we can define an embedding map \( E: [\emptyset, g] \rightarrow V^{\omega \lambda + 1} \) satisfying \( E([\emptyset, g]) = D(M) \) in natural way. Moreover, \( g < star \omega_1 \) since \( \mu_n < \omega_1 \cdot [\emptyset, g] \) is specially paracompact since \( \#(\omega_1) \) and \( [0, g] \) is closed in \( V^{\omega_1 + 1} \).

It implies \( D(M) \) is specially paracompact.

Suppose \( B \subseteq V^{\omega \lambda + 1} \). Remember that we have fixed \( f_x \in x \) for each \( x \in V^{\omega \lambda + 1} \). Let

\[
B_n = \{ f_x(n) : x \in B \}.
\]
If $B$ is countable, then the matrix
\[ M(B) = \bigcup_{n<\omega} \{(n,p): p \in \overline{B_n}\} \]
is also countable, where $\overline{B_n}$ denotes the closure of $B_n$ in $\lambda + 1$. Moreover, $D(M(B))$ is a closed set since $D(M(B)) = \bigcap_{n<\omega} \overline{B_n}$.

2.4 Definition. Suppose that $[a,b]$ is a basic set in $\nu^{\omega_\lambda} + 1$, $B \subset \nu^{\omega_\lambda} + 1$ is countable and $\mathcal{R}$ is an open cover of $[a,b]$. The notion of tops refinement of $\mathcal{R}$ relative to $B$ and $[a,b]$ is defined by the following cases:

Case 1. $D(M(B)) \cap [a,b] = \emptyset$. There is an open set $U \in \mathcal{R}$ such that $b \in U$. We choose a basic set $V$ satisfying $b \in V$ and $V \subset U \cap [a,b]$. In this case, the tops refinement is a singleton basic set $\{V\}$.

Case 2. $D(M(B)) \cap [a,b] \neq \emptyset$. Assume $\#(\omega_1)$. By 2.3, $D(M(B))$ is specially paracompact since $D(M(B))$ is closed. Then $D(M(B)) \cap [a,b]$ also is specially paracompact. Hence there is a refinement $\mathcal{U}$ of $\mathcal{R}$ consisting of pairwise disjoint basic sets which cover $D(M(B)) \cap [a,b]$. We call $\mathcal{U}$ tops refinement of $\mathcal{R}$ relative to $B$ and $[a,b]$.

Since a tops refinement might not cover $[a,b]$, we notice that a tops refinement need not be a refinement in the usual sense. However, we wish to make use of those sets not belonging to the tops refinement.

2.5 Lemma. Suppose $B \subset \nu^{\omega_\lambda} + 1$ is countable, $[a,b]$ is a basic set in $\nu^{\omega_\lambda} + 1$ and $\mathcal{R}$ is an open cover of $[a,b]$. Then there is a tops refinement $\mathcal{U}$ of $\mathcal{R}$ relative to $B$ and $[a,b]$, and there is a partition $\mathcal{P}$ of $[a,b]$ into basic sets such that $\mathcal{U} \subset \mathcal{P}$.
Proof. By 2.3 $D(M(B))$ is specially paracompact. Then $K = D(M(B)) \cap [a,b]$ is specially paracompact. For $K$, as a subspace of $\nu^\omega + 1$, there is a refinement $\mathcal{V}$ of $\{U \cap K: U \in \mathcal{R}\}$ consisting of pairwise disjoint basic sets in $K$. The tops refinement $\mathcal{U}$ of $\mathcal{R}$ relative to $B$ and $[a,b]$ can be induced by $\mathcal{V}$ in the following way.

First of all, we define a map from $\mathcal{V}$ into the family of all basic sets in $\nu^\omega + 1$. If $V = [r,s] \cap K \in \mathcal{V}$, $r,s \in K$, we define $\phi(V) = [\bar{r},s] \subset [a,b]$ by the following clauses:

1. $r(n) < s(n)$. Define $\bar{r}(n)$ from $r(n)$. If $r(n)$ is a successor ordinal, define

$$\bar{r}(n) = \sup\{p \in \lambda + 1: p < r(n) \& p \in K\} + 1$$

if $r(n)$ is a limit ordinal and $S = \{p \in \lambda + 1: p < r(n) \& p \in K\} \not= \emptyset$; define $\bar{r}(n) = a(n)$ if $r(n)$ is a limit ordinal and $S = \emptyset$.

2. $r(n) = s(n)$. In this case $s(n)$ is an isolated point of $K$. Define $\bar{r}(n) = s(n)$ if $s(n)$ is still an isolated point in $\lambda + 1$. If $s(n)$ is a limit ordinal, then we define $\bar{r}(n)$ in the same way as we did in (1).

We claim that:

(a) $V = \phi(V) \cap K$ for every $V \in \mathcal{V}$;

(b) $\phi(\mathcal{V}) = \{\phi(V): V \in \mathcal{V}\}$ is pairwise disjoint;

(c) $U\phi(\mathcal{V})$ is a basic set.

The clauses (a) and (b) are trivial. We prove (c).

In fact, let $h_n = \sup K_n$, then

$$[a,h] = U\phi(\mathcal{V}),$$

where $h = \chi_{\omega<\omega}h_n$. Let us prove this equality. Suppose $x \in [a,h]$. Then $a(n) \leq x(n) \leq h(n)$ for almost all $n$. Let
\[ m(n) = \min P, \text{ if } P = \{ \alpha \in K_n : \alpha > x(n) \} \neq \emptyset; \]
\[ m(n) = h(n), \text{ if } P = \emptyset; \quad m = (m(0), m(1), \ldots, m(n), \ldots). \]

Since \( m \in K \), there is a \( V = [r, s] \cap K \) such that \( m \in V \). Then
\[ \overline{r}(n) \leq r(n) \leq x(n) \leq m(n) \leq s(n) \]
according to (1), (2) and \( r, m, s \in K \). It means \( x \in \phi(V) \).
Then \( x \in U \phi(V) \). So far we have proved the inclusion "\( \subseteq \)".

The other inclusion "\( \supseteq \)" is trivial.

Notice that every \( \phi(V) \) contains only one member \( V \) of \( V \), and \( V \subset U \) for some \( U \in \mathcal{R} \). Without loss of generality, we suppose \( U \) is a basic set. Then we can find naturally a basic set \( \psi(V) \) in \( V^{\omega+1} \) such that
\[ V \subset \psi(V) \subset \phi(V) \text{ and } \psi(V) \subset U. \]

Let
\[ U = \{ \psi(V) : V \in V \}, \quad H = \{ \phi(V) \setminus \psi(V) : V \in V \} \cup \{ [a, b] \setminus [a, h] \} \]

Then
\[ [a, b] = [a, h] \cup ([a, b] \setminus [a, h]) = (uU) \cup (uH). \]

By 1.4, every member of \( H \) is a union of pairwise disjoint basic sets. Hence there is a collection \( \mathcal{G} \) of pairwise disjoint basic sets such that \( \cup \mathcal{G} = \cup H \). Then \( \mathcal{P} = U \cup \mathcal{G} \) is a partition of \( [a, b] \) and \( U \subset \mathcal{P} \).

2.6 Definition. Every element of \( \mathcal{P} \setminus U \) is called an uncovered tape.

3. The Proof of the Main Theorem

We assume that there is a \( \kappa \)-scale in \( \omega \). We are going to prove that \( \#(\lambda) \) is true for every ordinal \( \lambda \).
For simplicity, let \( X = \mathcal{V}^\omega \lambda + 1 \). Suppose \( \mathcal{R} \) is an open cover of \( X \). We intend to build a tree \( T \) consisting of basic sets (exactly, of uncovered tapes). The tree \( T \) is ordered by \( > \) and the height of \( T \) is \( \omega_1 \), where the order \( > \) is defined by

\[
[a, b] > [r, s] \iff a \preceq^* r \preceq^* s \preceq^* b
\]

(\( s \preceq^* b \) means \( s \preceq b \) and \( s \neq^* b \)). Simultaneously, we will construct a collection \( G_\alpha \) of basic sets for each ordinal \( \alpha < \omega_1 \) so that \( \mathcal{U} = \bigcup \{G_\alpha : \alpha < \omega_1\} \) is a refinement of \( \mathcal{R} \) covering \( X \). All of them subject to the following restrictions:

1. **(3.1)** The level 0, which is denoted by \( T_0 \), of \( T \) is \( \{X\} \) and \( G_0 = \emptyset \).

2. **(3.2)** \( T_\alpha \) denotes the \( \alpha \)'th level of \( T \). Then \( (\bigcup T_\alpha) \cup (\bigcup G_\alpha) = X \) and the elements of \( T_\alpha \cup G_\alpha \) are pairwise disjoint for every \( \alpha < \omega_1 \).

3. **(3.3)** For each \( \alpha < \omega_1 \) and \( V \in G_\alpha \) there is an \( U \in \mathcal{R} \) such that \( V \subseteq U \).

4. **(3.4)** The elements of \( \bigcup_{\alpha < \eta} G_\alpha \) are pairwise disjoint for all \( \eta < \omega_1 \).

5. **(3.5)** \( \alpha < \beta < \omega_1 \) implies \( G_\alpha \subseteq G_\beta \) (Then we have \( \bigcup (G_\beta \setminus G_\alpha) \subseteq \bigcup T_\alpha \), \( \alpha < \beta \), which follows from (3.2) and the inclusion \( G_\alpha \subseteq G_\beta \)).

6. **(3.6)** Suppose \( A \) is a branch of \( \eta \)'th subtree \( \bigcup \{T_\alpha : \alpha < \eta\} \), \( \eta < \omega_1 \), then the length of \( A \) is \( \eta \), and the intersection \( \cap A \) is non-empty.

Suppose \( \mu < \omega_1 \). We assume inductively that, for each \( \xi < \mu \), \( T_\xi \), \( G_\xi \), \( \alpha < \xi \) have been built and satisfy (3.1)-(3.6).
by taking $\xi$ instead of $\omega_1$ in the statements. It is easy to check that $T_\alpha$, $G_\alpha$, $\alpha < \mu$, also satisfy (3.1)-(3.6) by taking $\mu$ instead of $\omega_1$ in the statements.

Now we build $T_\mu$ and $G_\mu$ by the following way.

First of all, we claim that

$$\cap_{\alpha < \mu} (UT_\alpha) = U(\cap A: A \in \beta),$$

where $\beta$ is the collection of all branches of the $\mu$'th sub-tree $U(T_\alpha: \alpha < \mu)$. It follows trivially from (3.6) and (3.2).

Suppose $A = \{V_\alpha = [a_\alpha, b_\alpha] \in T_\alpha: \alpha < \mu\}$ is a branch. We conclude that $A = \cap A \neq \emptyset$, because there is no $(\omega, \omega^*)$ gap in $\omega$ and $\mu < \omega_1$. Moreover, the set $A$ is clopen since $\mu < \omega_1$ and $X$ is a P-space (by 1.1). Then $A = \cup S_A$ where $S_A$ is a collection of pairwise disjoint basic sets (by 1.4).

Let $B = \{b_\alpha: [a_\alpha, b_\alpha] \in A\}$. By 2.5 there is a tops refinement $U_C$ of $R$ relative to $B$ and $C, C \in S_A$, and there is a partition $P_C$ of $C$ such that $U_C \subseteq P_C$. Let

$$W = \{A = \cap A: A \in \beta\}.$$ We define

$$G_\mu = (\cup\{U_C: C \in S_A \& A \in W\}) \cup (\cup\{G_\alpha: \alpha < \mu\}),$$

$$T_\mu = \cup\{P_C \cap U_C: C \in S_A \& A \in W\}.\tag{2}$$

If $[r,s] \in T_\mu$, then $[r,s] < V_\alpha$ for all $V_\alpha \in A$. In fact, $[r,s] \in P_C$ for some $C \in S_A$ and some $A \in W$, then $[r,s] \subset A \subset V_\alpha$ for every $V_\alpha \in A$. On the other hand, $[r,s] \notin U_C$ implies $s \notin U_C$. But the top of $V_\alpha = [a_\alpha, b_\alpha]$ is an element of $U_C$. Hence $s \neq* b$.

The rest of the job is to check if $T_\alpha$, $G_\alpha$, $\alpha < \mu + 1$ still satisfy (3.1)-(3.6) by taking $\mu + 1$ instead $\omega_1$. We check the clause (3.2) and leave the rest to the readers.
It is easy to check that
\[(\bigcap_{\alpha<\mu} (\mathcal{U}_\alpha)) \cup \bigcup_{\alpha<\mu} (\mathcal{U}_\alpha) = \mathcal{X}. \quad (3)\]

Now we prove
\[(\mathcal{U}_\mu) \cup (\mathcal{U}_\mu) = \mathcal{X}, \]
i.e. (3.2) holds. In fact, if \(x \in \mathcal{X}\), then either
\[x \in \bigcap_{\alpha<\mu} (\mathcal{U}_\alpha) \text{ or } x \notin \bigcap_{\alpha<\mu} (\mathcal{U}_\alpha).\]
If \(x \in \bigcap_{\alpha<\mu} (\mathcal{U}_\alpha)\), then
\[x \in \bigcup (\cap A : A \in \mathcal{B})\]by the equality (1). Thus
\[x \in \bigcap A = A, \]
\[x \in C,\]
where \(A \in \mathcal{B}, C \in \mathcal{S}_A\) and \(A = \mathcal{U} S_A\). Because \(C = \mathcal{U} P_C\) and
\[U_C \subset P_C, \]
we have
\[x \in (\mathcal{U} (P_C \setminus U_C)) \cup (\mathcal{U} U_C).\]
So
\[x \in (\mathcal{U}_\mu) \cup (\mathcal{U}_\mu). \quad (4)\]
If \(x \notin \bigcap_{\alpha<\mu} (\mathcal{U}_\alpha)\), then the fact (4) follows from (3) and
(2).

The clauses (3.2), (3.3) imply that \(U = \mathcal{U} \{G : \alpha < \omega_1\}\)is a refinement of \(\mathcal{R}\) and the elements of \(U\) are pairwise disjoint. Is \(U\) a cover of \(\mathcal{X}\)? We have to prove the following theorem in order to answer the question.

3.7 Theorem. Suppose \(A = \{V_\alpha = [a_\alpha, b_\alpha] : \alpha \in T_\alpha : \alpha < \omega_1\}\)is a branch of the tree \(T\). Then \(\bigcap A = \emptyset\).

Proof. If the conclusion is false, then there is a point \(x \in \bigcap A\). Let \(B_\mu\), denote the set \(\{b_\alpha : \alpha < \mu\}\).
\[M(B_\mu)\]denote such a submatrix of \(M(B_\mu)\) that
\[M(B_\mu)\downarrow x = \{(n, j) \in M(B_\mu) : n < \omega, j > x(n)\}.
We say that a point \(b \in X\) extends a matrix \(M\) if there is an infinite set \(E \subset \omega\) such that \(b(n) < j\) for all \(n \in E\) and
\[(n, j) \in M.\]
We conclude that, for each \( \mu < \omega_1 \), \( b_\mu \) extends \( M(B_\mu) \) \( \upharpoonright x \).

In fact, since \([a_\mu, b_\mu] \in T_\mu\), there is a basic set
\( C \subseteq \bigcap\{ V_\alpha \in T_\alpha : \alpha < \mu \} \) such that
\[ [a_\mu, b_\mu] \subseteq \mathcal{P}_C \setminus U_C. \]

Because \( U_C \) covers \( D(M(B_\mu)) \) \( \cap C \), and \( \mathcal{P}_C \) is a partition of \( C \), we have
\[ [a_\mu, b_\mu] \cap D(M(B_\mu)) = \emptyset. \]

Hence
\[ [x, b] \cap D(M(B_\mu)) = \emptyset. \]  (5)

If the assertion fails then for every infinite \( E \subseteq \omega \) there is some \( n \in E \) and \( \xi \in (B_\mu)_n \) with \( x(n) < \xi < b(n) \). So we can find an \( m \in \omega \), for each \( n > m \), there is \( \xi_n \in (B_\mu)_n \) with \( x(n) < \xi_n < b(n) \). Let \( f(n) = \xi_n \) for every \( n > m \). Then
\[ f \in [x, b] \cap D(M(B_\mu)). \]

It is contradictory to (5).

There are only \( \omega_1 \) many \( b_\mu \)'s. So the extending will go \( \omega_1 \) many times. It is impossible. Why? Suppose
\( \mu_{n+1} < \omega_1 \) is an ordinal. We define inductively
\[ \mu_{n+1} = \begin{cases} \min\{ \eta : x(n) < b_\eta(n) < b_\mu(n) \}, & \text{if } \exists \eta > \mu_n \ (x(n) < b_\eta(n) < b_\mu(n)) \\ \mu_n, & \text{if } \not\exists \eta > \mu_n \ (x(n) < b_\eta(n) < b_\mu(n)) \end{cases} \]
and \( b_{\mu_0} = b_0 \). Then
\[ x(n) < \cdots < b_{n+1}(n) < \cdots < b_{\mu_1}(n) < b_{\mu_0}(n), \]  (*)
for every \( n < \omega \). There just are finitely many "<" appearing in the line (*) since every \( b_{\mu_j} \) is an ordinal. So there is a minimum among \( b_{\mu_j} \)'s (\( j = 0, 1, \ldots \)). Let
\[ b(n) = \min \{ b_{\mu_n}^j(n) : j < \omega \}, \]

\[ \mu_n = \min \{ \mu_{nJ} : b_{\mu_n}^j(n) = b(n) \}. \]

It is clear that for each \( n < \omega \) there is not any ordinal \( \eta > \mu_n \) such that

\[ x(n) < b_{\eta}^j(n) < b_{\mu_n}^j(n). \]

Let

\[ \gamma = \sup \{ \mu_n : n < \omega \}. \]

\( \gamma < \omega_1 \) since every \( \mu_n < \omega_1 \). If the extending goes \( \omega_1 \) many times, then \( b_{\gamma} \) extends \( M(B_{\gamma}) \). It implies that there is an infinite set \( E \subset \omega \) such that

\[ x(n) < b_{\gamma}^j(n) < b_{\mu_n}^j(n), \quad n \in E, \]

since every \( b_{\mu_n} \) is on the matrix \( M(B_{\gamma}) \). It is a contradiction.

It is similar to the equality (3) that the following equality holds

\[ \bigcap_{\alpha < \omega_1} (UT_{\alpha}) \cup (\bigcup_{\alpha < \omega_1} (UG_{\alpha})) = X. \]

The theorem 3.7 implies

\[ \bigcap_{\alpha < \omega_1} (UT_{\alpha}) = \emptyset. \]

So

\[ \bigcup_{\alpha < \omega_1} (UG_{\alpha}) = X. \]

i.e. \( \mathcal{U} \) covers \( X \).

References


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