REMARKS ON CLOSURE-PRESERVING SUM THEOREMS

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1. Introduction

In 1975 Junnila and Potoczny [6] and Katuta [2] independently showed that, if a space $X$ has a closure-preserving cover by compact sets, then $X$ is metacompact. Thus began a study of spaces which have closure-preserving covers by "nice" (finite, countably compact, Lindelof, etc.) sets. The reader is referred to [1,2,4,5,6,8,11,12,13,16,17,18,19,24] for these developments as well as their relationship with winning strategies in some topological games.

In 2 we show that if a space $X$ has a closure-preserving cover by nowhere dense sets, then $X$ need not be 1st category. However, if these sets are also countably compact, then $X$ must be 1st category. The closure-preserving property of the cover is strengthened in a natural way in 3, and a somewhat more general result is obtained for the ideal of closed subsets of a space. The desired result for nowhere dense sets follows as a special application. Finally in 4 closure-preserving sum theorems are obtained for the dimension functions dim and Ind for normal and totally normal spaces.

1. The authors would like to thank the referee for his/her comments on this paper.
Definition 1.1. (i) A family \( J \) of subsets of a space \( X \) is closure-preserving (c-p) if \( \cup \{ F : F \in J' \} = \overline{\cup J'} \) for every \( J' \subseteq J \).

(ii) A c-p family \( J \) is special if there exists a point finite open collection \( U \) such that for \( F \in J \),
\[
X - F = \cup \{ U \in U : U \cap F = \emptyset \}.
\]
In this case we say that \( U \) generates \( J \).

Definition 1.2. A family \( J \) of closed subsets of a space \( X \) is an ideal of closed sets if,

(i) for every finite \( J' \subseteq J \), \( \cup J' \in J \) and

(ii) if \( H \) is a closed set and \( H \subseteq J \in J \), then \( H \in J \).

We will denote the family of countable unions of members of \( J \) by \( \sigma J \).

Remark. As noted in [1], if \( U \) is a point finite open cover which generates a c-p \( J \)-cover, then the family
\[
C_n = \{ x_n \cap \bigcap U_n : x \in X \text{ and } |U_x| = n \}
\]
consists of \( J \)-small sets, and \( C_n \) is a discrete collection of relative closed sets in \( X_n = X_{n-1} \), where
\[
x_n = \{ x \in X : |U_x| < n \}.
\]

Definition 1.3. A space \( X \) is called weakly \( \mathcal{J} \)-refinable if every open cover of \( X \) has an open refinement \( \cup_{i=1}^{\infty} \mathcal{J}_i \) satisfying

(i) \( \{ G_i \}_{i=1}^{\infty} \) is point finite, where \( G_i = \cup \mathcal{J}_i \) and

(ii) for each \( x \in X \), there exists some \( n(x) \) such that
\[
0 < \text{ord}(x, \mathcal{J}_{n(x)}) < \infty.
\]
A cover $\bigcup_{i=1}^{\infty} C_i$ satisfying (i) and (ii) above is called a weak $\theta$-cover.

It is well known that the class of weakly $\theta$-refinable spaces lies strictly between the classes of $\theta$-refinable and weakly $\theta$-refinable spaces (see [8]).

All spaces will be assumed to be $T_2$.

2. Closure-preserving covers by nowhere dense sets

In this paper we consider spaces which have a closure-preserving cover by nowhere dense sets. Note that if $Q = \{r_1, r_2, \ldots\}$ is the set of rationals and $F_n = \{r_i\}_{i=1}^{n}$, then $\{F_i : i < \omega\}$ is a closure-preserving cover of $Q$ by finite sets. Clearly, $Q$ is a 1st category space.

The following example shows that if a space $X$ has a closure-preserving cover by nowhere dense sets, then $X$ need not be of the 1st category.

**Example.** Let $L = \{0, 1\}^\omega$ where $\{0, 1\}$ is the two point discrete space. The topology on $Y$ is the countable box topology; that is, a basic open set in $Y$ is a countable intersection of basic open sets in the usual product topology. Define $X = \{y \in Y : |\alpha < \omega_1 : y(\alpha) \neq 0 | < \omega_1\}$. Now if $F_\alpha = \{x \in X : x(\beta) = 0$ for each $\beta \geq \alpha\}$, it is easy to check that $\{F_\alpha : \alpha < \omega_1\}$ is a closure-preserving cover of $X$ by discrete closed sets. Furthermore each $F_\alpha$ is nowhere dense. We assert that $X$ is not 1st category. Indeed, let $\{C_n : n < \omega\}$ be a sequence of open dense subsets of $X$. Let $B_0$ be a basic open set such that $B_0 \subseteq C_0$, and define
\[ \mathcal{C}_0 = \{ \alpha < \omega_1 : x(\alpha) = y(\alpha) \text{ for all } x, y \in B_0 \}. \]

Now by induction choose a basic open set \( B_n \subset \mathcal{C}_n \cap B_{n-1} \) and let

\[ \mathcal{C}_n = \{ \alpha < \omega_1 : x(\alpha) = y(\alpha) \text{ for all } x, y \in B_n \}. \]

Since \( \mathcal{C} = \bigcup_{n<\omega} \mathcal{C}_n \) is countable, there exists a point \( z \in X \) such that \( z \in \bigcap_{n<\omega} B_n \subset \bigcap_{n<\omega} \mathcal{C}_n \) and \( z(\alpha) = 0 \) for \( \alpha > \sup C \). Therefore \( X \) is not 1st category.

The following result is an easy consequence of Zorn's Lemma so the proof is omitted.

**Lemma 2.1.** Let \( J = \{ F_\alpha : \alpha \in A \} \) be any non-empty family of non-empty sets. Then there exists a maximal pairwise disjoint subfamily of \( J \).

**Definition 2.2.** Let \( \phi \in E \subset X \) and \( J \) a family of subsets of \( X \).

1. \( m(E) = \) a maximal pairwise disjoint subfamily of \( \{ F \cap E : F \in J \} \).
2. \( M^*(E) = \bigcup m(E) \).

The following uses a technique due to Telgársy and Yajima [16: Theorem 3.3].

**Lemma 2.3.** Let \( \mathcal{C} = \{ C_\alpha : \alpha \in A \} \) be a closure-preserving cover of \( X \) by countably compact sets. If \( \{ H_n \}_{n=1}^{\infty} \) is any decreasing sequence of closed subsets of \( X \) satisfying \( H_{n+1} \subset X - M^*(H_n) \) for each \( n \), then \( \bigcap_{n=1}^{\infty} H_n = \phi \).

**Proof.** Suppose there exists a sequence \( \{ H_n \}_{n=1}^{\infty} \) of closed sets satisfying the above condition, but
\[ \cap_{n=1}^{\infty} H_n \neq \emptyset. \] Then there exists some \( C_{\alpha_0} \in \mathcal{C} \) such that \( C_{\alpha_0} \cap H_n \neq \emptyset \) for each \( n \). Furthermore \( (C_{\alpha_0} \cap H_n) \notin \mathcal{M}(H_n) \); for otherwise, \( C_{\alpha_0} \cap H_{n+1} = \emptyset \). Therefore for each \( n \), there exists some \( C_{\alpha_n} \in \mathcal{C} \) such that \( (C_{\alpha_n} \cap H_n) \in \mathcal{M}(H_n) \) and \( C_{\alpha_n} \cap (C_{\alpha_0} \cap H_n) \neq \emptyset \). Now choose \( x_n \in C_{\alpha_n} \cap (C_{\alpha_0} \cap H_n) \) so that \( x_n \in M^*(H_n) \) and hence \( x_n \notin H_{n+1} \). Then \( \{x_n\}_{n=1}^{\infty} \) is a sequence of distinct points in \( C_{\alpha_n} \) and must cluster at \( y \in C_{\alpha_0} \). Now \( y \in \{x_n: n \geq 1\} \subseteq \cup_{n=1}^{\infty} C_{\alpha_n} \) and \( y \in \cap_{n=1}^{\infty} H_n \) as well. But this is a contradiction, since \( y \in (C_{\alpha_n} \cap H_n) \in \mathcal{M}(H_n) \) implies that \( y \notin H_{n+1} \). Therefore it must be the case that \( \cap_{n=1}^{\infty} H_n = \emptyset \).

Remark. It should be noted that the above proof only used the fact that every countable subfamily of \( \mathcal{C} \) was closure-preserving.

Theorem 2.4. Let \( X \) be a regular space with a closure-preserving cover \( \mathcal{C} \) by countably compact nowhere dense sets. Then \( X \) is 1st category.

Proof. Since \( M^*(X) \) is closed and nowhere dense in \( X \), by Lemma 2.1 above, there exists a maximal pairwise disjoint family \( \mathcal{H}(X) \) of regular closed subsets of \( X \) such that \( \cup \mathcal{H}(X) \subseteq X - M^*(X) \). Furthermore, it is easy to see that \( \cup \mathcal{H}(X) \) is dense in \( X \). Define \( \mathcal{G}_0 = \{\text{int}(H): H \in \mathcal{H}(X)\} \) and \( G_0 = \cup \mathcal{G}_0 \). Note that \( G_0 \) is also dense in \( X \). Now by induction we construct the sequence \( (\mathcal{G}_0, G_0, \mathcal{G}_1, G_1, \ldots, \mathcal{G}_n, G_n, \ldots) \) where

\[ \mathcal{G}_{n+1} = \{\text{int}(H): H \in \mathcal{H}(G) \text{ where } G \in \mathcal{G}_n\}, \]

\[ G_{n+1} = \cup \mathcal{G}_{n+1} \text{ and } \]
\( \mathcal{H}(\mathcal{G}) \) = a maximal pairwise disjoint family of regular closed subsets of \( X \) such that \( \bigcup \mathcal{H}(\mathcal{G}) \) is dense in \( \mathcal{G} = M^*(\mathcal{G}) \).

As before it is easy to see that \( G_n \) is open and dense in \( X \).

Finally, if \( H_0 = X \) and \( H_{n+1} \in \mathcal{H}(H_n) \), by Lemma 2.3, \( \bigcap_{n=0}^{\infty} H_n = \emptyset \). Therefore \( \bigcap_{n=0}^{\infty} G_n = \emptyset \) and hence \( X \) is 1st category.

Corollary 2.5. Let \( X \) be a regular countably compact space. If \( \mathcal{J} \) is a closure-preserving cover of \( X \), then some member of \( \mathcal{J} \) has non-empty interior.

3. Special c-p covers

In [1] the authors studied spaces having c-p \( \mathcal{J} \)-covers induced by a point finite open cover. Here we get analogous results to Theorem 2.4 above by omitting the regularity and countably compact conditions.

The notion of \( \mathcal{B}(D,L) \)-refinability has been shown to play an important role in the study of weakly \( \mathcal{G} \)-refinable spaces introduced in [7].

Definition 3.1. A space \( X \) is \( \mathcal{B}(D,L) \)-refinable provided every open cover \( \mathcal{U} = \{ U_\gamma : \gamma \in \Gamma \} \) of \( X \) has a refinement \( \mathcal{E} = \bigcup \mathcal{E}_\beta = \{ E(\beta, \gamma) : \gamma \in \Gamma \} : \beta < \lambda \) where satisfies

(i) \( E(\beta, \gamma) \subseteq U_\gamma \) for each \( \gamma \in \Gamma \), \( \beta < \lambda \),
(ii) \( \{ U_\beta : \beta < \lambda \} \) partitions \( X \),
(iii) for each \( \beta < \lambda \), \( \mathcal{E}_\beta \) is a closed discrete collection in \( X = \bigcup \{ U_\mu : \mu < \beta \} \), and
(iv) for each \( \beta < \lambda \), \( \bigcup \{ U_\mu \mathcal{E}_\mu : \mu < \beta \} \) is closed in \( X \).
In [7] the author has shown that every space which is $B(D,\omega)$-refinable is weakly $\omega$-refinable and that every weakly $\omega$-refinable space is $B(D,(\omega)^2)$-refinable. Throughout this paper $\lambda$ will denote a countable ordinal.

**Definition 3.2.** Let $J$ be an ideal of closed subsets of a space $X$. We say that $J$ is **discretely additive** if 
(i) $J$ is closed under discrete unions, and 
(ii) when $F \in \sigma J$ and $\emptyset \subseteq J$ such that $\emptyset$ is discrete in $X - F$, then $H = F \cup \{\emptyset\} \in \sigma J$.

**Theorem 3.3.** If a space $X$ has a $B(D,\lambda) - J$ cover for some countable ordinal $\lambda$ and $J$ is discretely additive, then $X \in \sigma J$.

**Proof.** Let $\mathcal{E} = \bigcup \{\mathcal{E}_\beta : \beta < \lambda\}$, where $\mathcal{E}_\beta = \{E(\beta,\gamma) : \gamma \in \Gamma\}$, be a $B(D,\lambda) - J$ cover of $X$ such that $J$ is discretely additive. The proof is by induction on $\lambda$. It is easy for $\lambda = 1$; since $J$ is closed under discrete unions, so $X \in J$. Let $\beta < \lambda$ and assume true for all $\gamma < \beta$. If $\beta = \alpha_0 + 1$, define $F = \bigcup \{\mathcal{E}_\mu : \mu < \alpha_0\}$ so that $F \in \sigma J$.

Now $\mathcal{E}_\beta$ is discrete in $X - F$ and $\mathcal{E}_\beta \subseteq J$, and hence $H_\beta = F \cup \{\mathcal{E}_\beta\} \in \sigma J$. If $\beta$ is a limit ordinal, let $\alpha_i < \beta$ such that $\alpha_i < \alpha_{i+1}$ and $\sup \{\alpha_i\} = \beta$. Now for each $i$, $H_i = \bigcup \{\mathcal{E}_\mu : \mu < \alpha_i\} \in \sigma J$, so $H_\beta = \bigcup_{i=1}^{\omega} H_i \in \sigma J$. Therefore $X \in \sigma J$.

We now consider the ideal of closed nowhere dense subsets of a space $X$. 
Lemma 3.4. Let $J$ be the ideal of closed nowhere dense subsets of a space $X$. Then $J$ is discretely additive.

Proof. Clearly, $J$ is closed under discrete unions. Let $F \in \mathcal{O}J$ and $D = \{D_{\alpha}: \alpha \in \Lambda\} \subseteq J$ such that $D$ is discrete in $X - F$. Let $W$ be any non-empty open subset of $X$ such that $W \subseteq H = F \cup (\cup D)$. If $U = \text{int}(F)$, then $W \subseteq U$; otherwise, $W \cap D_{\alpha} \neq \emptyset$ for some $\alpha \in \Lambda$. Thus $D_{\alpha}$ fails to be nowhere dense in $X$. Therefore, $\cup\{(D_{\alpha} - U): \alpha \in \Lambda\} \in J$; and hence $H \in \mathcal{O}J$.

Corollary 3.5. Let $X$ be a space with $B(D,\lambda)$-cover by nowhere dense sets. Then $X$ is $1^{st}$ category.

Proof. Let $J$ be the ideal of closed nowhere dense subsets of $X$. Then $J$ is discretely additive by Lemma 3.4, and hence $X \in \mathcal{O}J$ by Theorem 3.3.

Corollary 3.6. If a space $X$ has a special $c$-$p$ cover by nowhere dense sets, then $X$ is $1^{st}$ category.

4. Applications to Dimension Theory

In [24] the author obtained the following result which also can be found using a technique similar to that of Theorem 2.4 above.

Theorem 4.1. [24]. Let $X$ be a normal space. If $X$ has a $c$-$p$ cover $C$ by countably compact sets such that $\dim(C) \leq n$ for $C \in C$, then $\dim(X) \leq n$. 
To obtain the special closure-preserving sum theorem for covering dimension as an application of Theorem 3.6 above, we need the following result found in [24].

**Lemma 4.2.** Let $X$ be a normal space and $F$ a closed subset of $X$ with $\dim(F) \leq n$. For each finite open cover $\mathcal{U} = \{U_i: i \leq m\}$ of $X$, there exists an open refinement $\mathcal{V} = \{V_i: i \leq m\}$ of $\mathcal{U}$, and an open set $G \supset F$ such that

(i) $V_i \subset U_i$ for each $i \leq m$, and

(ii) $\text{ord}(x, V) < n + 1$ for each $x \in G$.

**Lemma 4.3.** Let $X$ be a normal space and $F$ a closed subset of $X$ with $\dim(F) \leq n$. Let $\mathcal{D} = \{D_\alpha: \alpha \in \Lambda\}$ be a family of closed subsets of $X$ such that

(i) $\dim(D_\alpha) \leq n$ for each $\alpha \in \Lambda$, and

(ii) $\mathcal{D}$ is discrete in $X - F$.

Then $\dim(F \cup \bigcup \mathcal{D}) \leq n$.

**Proof.** Let $\mathcal{U} = \{U_i: i \leq m\}$ be a finite open (in $X$) cover of $H = F \cup (\bigcup \mathcal{D})$. By Lemma 4.2 there exists an open refinement $\mathcal{V} = \{V_i: i \leq m\}$ of $\mathcal{U}$ and an open set $G \supset F$ such that $\text{ord}(x, V) < n + 1$ for all $x \in G$. Now $\{D_\alpha - G: \alpha \in \Lambda\}$ is a discrete closed collection in $X$ with $\dim(D_\alpha - G) \leq n$ for each $\alpha \in \Lambda$. Therefore $\dim(\bigcup (D - G)) \leq n$ and hence $\mathcal{V}$ has a partial open refinement $\mathcal{W} = \{W_i: i \leq m\}$ covering $\{\bigcup (D - G)\}$, such that $W_i \subset V_i$ and $\text{ord}(x, W) < n + 1$ for $x \in (\bigcup (D - G))$. Define $W_i^* = W_i \cup (V_i - G)$ for $i \leq m$. It is easy to show that $W^* = \{W_i^*: i \leq m\}$ is an open refinement of $\mathcal{U}$ covering $H$ and $\text{ord}(x, W^*) < n + 1$ for $x \in H$. Therefore $\dim(H) \leq n$. 
We now obtain the desired $c$-$p$ sum theorem for covering dimension.

**Theorem 4.4.** Let $X$ be a normal space which has a $B(D,\lambda)$ cover $\mathcal{E}$ such that $\dim(E) \leq n$ for each $E \in \mathcal{E}$. Then $\dim(X) \leq n$.

**Proof.** Let $J$ be the ideal of closed subsets of $X$ such that $\dim(J) \leq n$ for $J \in J$. Since covering dimension satisfies the countable sum theorem for normal spaces, $J = \sigma J$. Therefore $J$ is discretely additive by Theorem 4.3 above. Now from Theorem 3.3 it follows that $X \in \sigma J = J$. Hence $\dim(x) \leq n$.

**Corollary 4.5.** Let $X$ be a normal space. If $X$ has a special $c$-$p$ cover $\mathcal{C}$, such that $\dim(C) \leq n$ for each $C \in \mathcal{C}$, then $\dim(X) \leq n$.

For totally normal spaces similar results hold for large inductive dimension using the lemma below. The proofs are straightforward and left for the reader.

Yajima [18] has obtained a $\sigma$-closure-preserving sum theorem for Ind.

**Lemma 4.6.** Let $X$ be totally normal and $F$ a closed subset of $X$ such that $\text{Ind}(F) \leq n$. If $\text{Ind}(K) \leq n$ for each closed subset $K$ of $X$ such that $K \cap F = \emptyset$, then $\text{Ind}(K) \leq n$.

**Theorem 4.7.** Let $X$ be a totally normal space. If $X$ has a special $c$-$p$ cover $\mathcal{C}$ such that $\text{Ind}(C) \leq n$ for $C \in \mathcal{C}$, then $\text{Ind}(X) \leq n$. 
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