SPACES OF CONTINUOUS LINEAR FUNCTIONALS: SOMETHING OLD AND SOMETHING NEW

by

S. Kundu
SPACES OF CONTINUOUS LINEAR 
FUNCTIONALS: SOMETHING OLD 
AND SOMETHING NEW 

S. Kundu

Let $C(X)$ denote the set of all continuous real-valued functions on a completely regular Hausdorff space $X$ and $C^*(X)$ be the set of bounded functions in $C(X)$. Let us denote by $C_k(X)$ (respectively by $C_p(X)$) the set $C(X)$ topologized with the compact-open (respectively the point-open) topology. Both $C_k(X)$ and $C_p(X)$ are locally convex spaces. The locally convex compact-open topology on $C(X)$ is generated by the collection of seminorms $\{p_K: K \text{ is a compact subset of } X\}$ where $p_K(f) = \sup \{|f(x)|: x \in K\}$ for $f \in C(X)$. Similarly the locally convex point-open topology on $C(X)$ is generated by the collection of seminorms $\{p_F: F \text{ is a finite subset of } X\}$ where $p_F(f) = \sup \{|f(x)|: x \in F\}$. Let $K(X) = \{K \subseteq X: K \text{ is a compact subset of } X\}$ and $F(X) = \{F \subseteq X: F \text{ is a finite subset of } X\}$.

Basic open sets in $C_k(X)$ (respectively in $C_p(X)$) look like $<f,A,\varepsilon> = \{g \in C(X): |f(x) - g(x)| < \varepsilon \text{ for all } x \in A\}$ where $f \in C(X)$, $A \in K(X)$ (respectively $A \in F(X)$) and $\varepsilon > 0$.

Let $\Lambda_k(X)$ (respectively $\Lambda_p(X)$) be the set of all continuous linear functionals (real-valued functions) on $C_k^*(X)$ (on $C_p^*(X)$ respectively). Note since $C_k^*(X)$ (respectively $C_p^*(X)$) is a dense linear subspace of the
locally convex space $C_k(X)$ (respectively $C_p(X)$), the set of all continuous linear functionals on $C_k(X)$ (respectively $C_p(X)$) equals the set of all continuous linear functionals on $C_k(X)$ (respectively on $C_p(X)$). In [8], a normed linear space whose underlying set is $\Lambda_k(X)$ has been studied in detail. In [8], the notation $\Lambda(X)$ has been used in place of $\Lambda_k(X)$. A necessary condition for this normed linear space $\Lambda_k(X)$ to be complete is that $C(X) = C^*(X)$, that is, every real-valued continuous function on $X$ must be bounded. In this paper, we want to put the problem of completeness of $\Lambda_k(X)$ in a proper perspective and we show that the problem of completeness of $\Lambda_k(X)$ is essentially a problem of finding a suitable topology on $C^*(X)$. Because of the discussion in this paragraph, from now on, we will be interested only in $C^*(X)$. We want to answer the problem of completeness of $\Lambda_k(X)$ in a more general setting. For this purpose, we first define a new topology on $C^*(X)$ and we will see that the point-open, compact-open and sup-norm topologies on $C^*(X)$ are all special cases of this topology.

1. A New Topology on $C^*(X)$

Let $\alpha$ be a collection of subsets of $X$ which satisfies the following two conditions: (i) each member of $\alpha$ is $C^*$-embedded and (ii) if $A, B \in \alpha$, then there exists $C \in \alpha$ such that $A \cup B \subseteq C$.

For each $A \in \alpha$, define a seminorm $p_A$ on $C^*(X)$ as follows. For $f \in C^*(X)$, $p_A(f) = \sup \{|f(x)| : x \in A\}$. 
Consider the locally convex topology on $\mathcal{C}^*(X)$ generated by the collection of seminorms $\{p_A: A \in \alpha\}$. Because of (ii), for each $f \in \mathcal{C}^*(X)$, $f + U = \{f + V: V \in U\}$ is a neighborhood base at $f$ where $U = \{V_{p_A, \varepsilon}: A \in \alpha, \varepsilon > 0\}$.

We call this new locally convex topology on $\mathcal{C}^*(X)$ $\alpha$-topology and the corresponding topological space we denote by $\mathcal{C}^*_\alpha(X)$. Note when $\alpha = K(X)$ or $F(X)$, we get compact-open or point-open topology on $\mathcal{C}^*(X)$ respectively.

The supremum norm on $\mathcal{C}^*(X)$ is defined as $\|f\|_\infty = \sup \{|f(x)|: x \in X\}$ for $f \in \mathcal{C}^*(X)$. This supremum norm generates a finer topology than the $\alpha$-topology on $\mathcal{C}^*(X)$.

We denote this normed linear space by $\mathcal{C}^*_\infty(X)$. If $\alpha$ contains $X$, then $\mathcal{C}^*_\alpha(X) = \mathcal{C}^*_\infty(X)$; and if, in addition, we assume the members of $\alpha$ to be closed, then $\mathcal{C}^*_\alpha(X) = \mathcal{C}^*_\infty(X)$ only if $\alpha$ contains $X$. (see [7], page 7).

Let $\Lambda^*_\alpha(X)$ be the set of all continuous linear functionals (real-valued) on $\mathcal{C}^*(X)$ and let $\Lambda^*_\infty(X)$ be the set of all continuous linear functionals (real-valued) on $\mathcal{C}^*_\infty(X)$. Since the sup-norm topology on $\mathcal{C}^*(X)$ is finer than the $\alpha$-topology on it, $\Lambda^*_\alpha(X) \subset \Lambda^*_\infty(X)$. Now $\Lambda^*_\infty(X)$ is a normed linear space with the usual conjugate norm, that is, given $\lambda \in \Lambda^*_\infty(X)$, we have a norm $\|\lambda\|_* = \sup \{|\lambda(f)|: f \in \mathcal{C}^*(X), \|f\|_\infty \leq 1\}$ where $\|\cdot\|_\infty$ is the sup-norm on $\mathcal{C}^*(X)$.

Consequently we can assign this $\|\cdot\|_*$-norm on $\Lambda^*_\alpha(X)$ to make it a normed linear space $(\Lambda^*_\alpha(X), \|\cdot\|_*)$.

Note $\Lambda^*_\infty(X)$ is actually a particular case of $\Lambda^*_\alpha(X)$. Here we also mention another particular $\Lambda^*_\alpha(X)$. Let $X$ be
a normal Hausdorff space and $\sigma = \{\text{cl}_X A : A \text{ is a } \sigma\text{-compact subset of } X\}$. Note that $\sigma$ is closed under finite union because $\bigcup_{n=1}^k \text{cl}_X A_n = \text{cl}_X (\bigcup_{n=1}^k A_n)$. We denote the corresponding $\Lambda_\sigma(X)$ by $\Lambda_0(X)$. While considering $\Lambda_0(X)$, we will always assume $X$ to be a normal Hausdorff space.

2. Basic Properties of $\Lambda_\alpha(X)$

Let $\Lambda_\alpha^+(X) = \{\lambda \in \Lambda_\alpha(X) : \lambda \geq 0\}$ where $\lambda \geq 0$ provided that $\lambda(f) \geq 0$ for each $f \in C^*(X)$ such that $f \geq 0$. If $\lambda \in \Lambda_\alpha(X)$ and $A$ is a subset of $X$, then $\lambda$ is said to be supported on $A$ provided that whenever $f \in C^*(X)$ with $f|_A = 0$, then $\lambda(f) = 0$. Since $\lambda$ is linear, this is equivalent to saying that whenever $f, g \in C^*(X)$ with $f|_A = g|_A$, then $\lambda(f) = \lambda(g)$.

The next two lemmas can be proved in manners similar to Lemmas 1.1 and 1.2 in [8].

**Lemma 2.1.** For each $\lambda \in \Lambda_\alpha(X)$, there exists an element $A$ in $\alpha$ such that $\lambda$ is supported on $A$. Conversely, if $\lambda$ is a positive linear functional on $C^*(X)$ which is supported on an element of $\alpha$, then $\lambda \in \Lambda_\alpha^+(X)$.

**Lemma 2.2.** Let $A$ be a closed subset of $X$, let $F \in \alpha$ and let $\lambda \in \Lambda_\alpha(X)$. If $\lambda$ is supported on each of $A$ and $F$, then $\lambda$ is supported on $A \cap F$.

Now on $\Lambda_\alpha^+(X)$ we give a topology induced by the metric $d_*(\lambda, \mu) = \|\lambda - \mu\|$ for $\lambda, \mu \in \Lambda_\alpha^+(X)$. 
Theorem 2.3. \((\Lambda_{\alpha}^{+}(X), d_{\ast})\) is a closed subspace of \((\Lambda_{\alpha}(X), \| \cdot \|_{\ast})\).

Proof. Let \(\lambda \in \Lambda_{\alpha}(X) \backslash \Lambda_{\alpha}^{+}(X)\). Then there exists a \(g \in C^{\ast}(X)\) such that \(g \geq 0\) and \(\lambda(g) < 0\). Let \(r\) be a positive number such that \(r g \leq 1\). Define \(\varepsilon = \frac{r}{2} \lambda(g)\). Now suppose \(u \in \Lambda_{\alpha}(X)\) is such that \(\| u - \lambda \|_{\ast} < \varepsilon\). Then

\[ |u(g) - \lambda(g)| < \varepsilon \] so that \(u(g) - \lambda(g) < \frac{\varepsilon}{r} = -\frac{1}{2}\lambda(g)\).

Therefore \(u(g) < -\frac{1}{2}\lambda(g) < 0\) so that \(u \in \Lambda_{\alpha}(X) \backslash \Lambda_{\alpha}^{+}(X)\).

3. The Completeness of \(\Lambda_{\alpha}^{+}(X)\) and \(\Lambda_{\alpha}(X)\)

The space \(\Lambda_{\alpha}^{+}(X)\) is a metric space with the metric \(d_{\ast}\). This space is complete provided that if a sequence in \(\Lambda_{\alpha}^{+}(X)\) is a Cauchy sequence with respect to \(d_{\ast}\), then it converges. Likewise the normed linear space \(\Lambda_{\alpha}(X)\) is complete if it is complete with respect to its norm \(\| \cdot \|_{\ast}\), that is, if it is a Banach space.

We have studied the completeness of \(\Lambda_{K}(X)\) and \(\Lambda_{K}^{+}(X)\) in [8]. We already know that \(\Lambda_{\infty}(X)\), being the conjugate space of a normed linear space, is always complete.

To establish that the completeness of \(\Lambda_{\alpha}^{+}(X)\) is equivalent to the completeness of \(\Lambda_{\alpha}(X)\), we need the following theorem which can be proved like Theorem 2.2 in [8].

Theorem 3.1. Each \(\lambda \in \Lambda_{\alpha}(X)\) can be written as

\(\lambda = \lambda^{+} - \lambda^{-}\) where \(\lambda^{+}\) and \(\lambda^{-}\) are members of \(\Lambda_{\alpha}^{+}(X)\). Furthermore, if \(\lambda, \mu \in \Lambda_{\alpha}(X)\), then \(\| \lambda^{+} - \mu^{+} \|_{\ast} \leq \| \lambda - \mu \|_{\ast}\) and

\(\| \lambda^{-} - \mu^{-} \|_{\ast} \leq \| \lambda - \mu \|_{\ast}\).
Theorem 3.2. The metric space $\Lambda_+^a(X)$ is complete if and only if the normed linear space $\Lambda_+^a(X)$ is complete.

Proof. Use Theorems 3.1 and 2.3.

Because of Theorem 3.2, each of the following theorems about $\Lambda_+^a(X)$ is also true for $\Lambda_+^a(X)$.

Theorem 3.3. Suppose $X$ is infinite and $F(X) \subseteq a$. Now if $\Lambda_+^a(X)$ is complete, then every countable subset of $X$ is contained in some member of $a$.

Proof. Let $A = \{x_n : n \in \mathbb{N}\}$ be any countable subset of $X$. For each $m \in \mathbb{N}$, define $\lambda_m : C^*(X) \to \mathbb{R}$ as follows. For each $f \in C^*(X)$, take $\lambda_m(f) = \sum_{n=1}^{m} \frac{1}{2n} f(x_n)$. Each $\lambda_m$ is a positive linear functional on $C^*(X)$ supported on the finite set $\{x_1, \ldots, x_m\}$. Then by Lemma 2.1, $\lambda_m$ is continuous. Now for each $k$ and $m$ with $k < m$, $d_*(\lambda_k, \lambda_m) = 1 |\lambda_k - \lambda_m|_* \leq \sum_{n=k+1}^{m} \frac{1}{2n}$. Therefore $(\lambda_m)$ is a Cauchy sequence in $\Lambda_+^a(X)$. Since $\Lambda_+^a(X)$ is complete, the $(\lambda_m)$ converges to some $\lambda$ in $\Lambda_+^a(X)$. Also $\lambda_m + \lambda$ implies $\lambda(f) = \lim_{m \to \infty} \lambda_m(f) = \sum_{n=1}^{\infty} \frac{1}{2n} f(x_n)$ for all $f \in C^*(X)$.

Now suppose $\lambda$ has a support $Y$ which belongs to $a$. We show that $A \subseteq Y$. Suppose not, then there is some $m$ such that $x_m \not\in Y$. Since $X$ is completely regular, there is some continuous function $f$ on $X$ with values in the unit interval $I$ such that $f(x_m) = 1$ and $f(Y) = \{0\}$. Since $\lambda$ is supported on $Y$, $\lambda(f) = 0$. But $\lambda(f) = \sum_{n=1}^{\infty} \frac{1}{2n} f(x_n) \geq \frac{1}{2m} f(x_m) = \frac{1}{2m} > 0$. With this contradiction, it follows that $A \subseteq Y$.

Corollary 3.4. If $X$ is infinite, then $\Lambda_+^p(X)$ and $\Lambda_+^p(X)$ are not complete.
Theorem 3.5. If the closure of each countable union of elements of \( \alpha \) belongs to \( \Lambda^+_\alpha(X) \), then \( \Lambda^+_\alpha(X) \) is complete.

Proof. Let \( (\lambda_n) \) be a Cauchy sequence in \( \Lambda^+_\alpha(X) \). Consider \( \Lambda^+_\alpha(X) \) as a subspace of the complete metric space \( \Lambda^+_\omega(X) \). Then \( (\lambda_n) \) is a Cauchy sequence in \( \Lambda^+_\omega(X) \) and hence converges to some \( \lambda \) in \( \Lambda^+_\omega(X) \). Suppose each \( \lambda_n \) is supported on \( A_n \) where \( A_n \in \alpha \). We show that \( \lambda \) is supported on \( A = \text{cl}_X(\bigcup_{n=1}^{\infty} A_n) \). Let \( f \in C^*(X) \) with \( f|_A = 0 \). Since each \( \lambda_n \) is supported on \( A_n \subseteq A \), then each \( \lambda_n(f) = 0 \) and consequently \( \lambda(f) = \lim_{n \to \infty} \lambda_n(f) = 0 \). Therefore \( \lambda \) has support \( A \). But by hypothesis \( A \in \alpha \). Hence by Lemma 2.1 \( \lambda \in \Lambda^+_\alpha(X) \). So \( \Lambda^+_\alpha(X) \) is complete.

Corollary 3.6. Suppose \( X \) is a normal Hausdorff space. Then \( \Lambda^+_\emptyset(X) \) is always complete.

Proof. Suppose for each \( n \), \( A_n \) is a \( \sigma \)-compact subset of \( X \). Then \( \text{cl}_X(\bigcup_{n=1}^{\infty} \text{cl}_X A_n) = \text{cl}_X(\bigcup_{n=1}^{\infty} A_n) \in \sigma \).

4. Measure Theoretic-Counterparts

In this section, we will talk about the measure-theoretic counterparts of \( \Lambda^+_\alpha(X) \) and \( \Lambda^+_\omega(X) \) with some extra conditions on \( \alpha \) and \( X \). So now we introduce some ideas from measure theory.

The algebra generated by the closed sets of \( X \) are denoted by \( \Lambda^+_\emptyset \) while the \( \sigma \)-algebra they generate is denoted by \( \sigma \), called the Borel sets.

For us a finitely additive measure (also called signed measure) on \( \Lambda^+_\emptyset \) is a real-valued function defined
on $A_c$ satisfying the following two properties (i) $\mu(\emptyset) = 0$; (ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A, B \in A_c$ and $A \cap B = \emptyset$.

A finitely additive measure $\mu$ is called a countably additive measure or simply a measure provided that (iii) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for all pairwise disjoint sequences $(A_n)_{n=1}^{\infty}$ such that $A_n \in A_c$ and $\bigcup_{n=1}^{\infty} A_n \in A_c$.

When a measure $\mu$ is defined on $\mathcal{B}$, we call it a Borel measure. A measure $\mu$ defined on $\mathcal{B}$ has support $A$ where $A \subseteq X$ and $A \in \mathcal{B}$ if $|\mu|(X \setminus A) = 0$. A finitely additive measure $\mu$ defined on $A_c$ or $\mathcal{B}$ is regular whenever $A$ is in the domain of definition of $\mu$ and $\varepsilon > 0$, there are closed and open sets $C$ and $U$ such that $C \subseteq A \subseteq U$ and $|\mu|(U \setminus C) < \varepsilon$.

Note when $\mu$ has compact support, this definition of regularity coincides with the one usually given in the books on measure theory. For more information on measure theory see [4] and [6].

Now we fix some notations.

A (signed) measure $\mu$ defined on $\mathcal{B}$ is said to be a finite (signed) measure if $|\mu(A)| < \infty$ holds for each $A \in \mathcal{B}$. It can be shown that a signed measure $\mu$ is finite if and only if $|\mu|(X) < \infty$. So a finite signed measure defined on $\mathcal{B}$ has finite total variation. For details on the above, see [1], 26.

Now let $M_b^+(X)$ be the set of all finite (signed) regular Borel measures on $X$. Let $M_b^+(X) = \{\mu \in M_b(X): \mu \geq 0, \text{ that is, } \mu \text{ is a positive measure}\}$. Throughout the remaining part of this paper we will assume the following extra condition on $\alpha$: the members of $\alpha$ are closed.
Now define $M_{b,a}(X) = \{ \mu \in M_b(X): \mu \text{ has a support } A(\subseteq X) \text{ such that } A \in a \}$. Let $M_{b,a}^+(X) = \{ \mu \in M_{b,a}(X): \mu \geq 0 \}$. When $a = K(X)$ or $F(X)$, we write $M_{b,k}(X)$ or $M_{b,p}(X)$ respectively.

The next thing to observe is that given $\mu \in M_b(X)$, $\|\mu\| = |\mu|\mu(X)$ defines a norm on $M_b(X)$. So $(M_b(X),\|\cdot\|)$ is actually a normed linear space. Also $M^+_b(X)$ is a metric space when equipped with the norm $p$ given by $p(\mu_1,\mu_2) = \|\mu_1 - \mu_2\|$ for every $\mu_1,\mu_2 \in M_b^+(X)$. Note $(M_{b,a}(X),\|\cdot\|)$ is a normed linear space while $(M_{b,a}^+(X),p)$ is a metric space.

Before having our first theorem in this section, we need the following two lemmas.

**Lemma 4.1.** Suppose $Y$ is a Borel subset of a completely regular Hausdorff space $X$. Let $B(X)$ and $B(Y)$ be the $\sigma$-algebras of Borel subsets of $X$ and $Y$ respectively. Then $B(X) \cap Y = B(Y)$ where $B(X) \cap Y = \{ B \cap Y: B \in B(X) \}$.

**Proof.** Define $\mathcal{V} = \{ A \in P(X): A = E \cup (B \setminus Y); E \in B(Y) \}$ and $B \in B(X) \}$. where $P(X)$ is the power set of $X$. Note $X \setminus (E \cup (B \setminus Y)) = (Y \setminus E) \cup ((X \setminus (B \setminus Y)) \setminus Y)$. Now it can be easily shown that $\mathcal{V}$ is a $\sigma$-algebra on $X$ containing all the closed subsets of $X$. Hence $B(X) \subseteq \mathcal{V}$. So $B(X) \cap Y \subseteq \mathcal{V} \cap Y$. But $\mathcal{V} \cap Y = B(Y)$. So $B(X) \cap Y \subseteq B(Y)$. Note $B(X) \cap Y$ is a $\sigma$-algebra on $Y$ and if $C$ is a closed subset of $Y$, then $C = C' \cap Y$ for some closed subset $C'$ of $X$ which means $C \in B(X) \cap Y$. Hence $B(Y) \subseteq B(X) \cap Y$. Therefore $B(X) \cap Y = B(Y)$. 
Lemma 4.2. If $A$ is a compact subset of a completely regular Hausdorff space $X$, then for every closed set $B \subseteq X \setminus A$, there exists a continuous function $f: X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.

Proof. See [5], page 168.

Theorem 4.3. Suppose $\alpha \subseteq K(X)$, that is, the members of $\alpha$ are compact. Then $(M_{b,\alpha}^+(X), \|\cdot\|)$ is isometrically isomorphic to $(\Lambda_{\alpha}(X), \|\cdot\|_*)$ while $M_{b,\alpha}^+(X)$ is identified with $\Lambda_{\alpha}(X)$ under this isometric isomorphism.

Proof. Define $F: M_{b,\alpha}^+(X) \to \Lambda_{\alpha}(X)$ by $F(\mu)(f) = \int f \, d\mu$ for each $\mu \in M_{b,\alpha}^+(X)$ and $f \in C(X)$. Let $K$ be a compact support of $\mu$ belonging to $\alpha$, that is, $|\mu|(X \setminus K) = 0$ and $K \subseteq \alpha$. Then for each $f \in C(X)$, $|F(\mu)(f)| = \left| \int f \, d\mu \right| = \left| \int_K f \, d\mu \right| \leq \int_K |f| \, d|\mu| \leq |\mu|(K) \cdot p_K(f)$ and so $F(\mu)$ is continuous. Clearly $F(\mu)$ is linear. Hence $F(\mu) \in \Lambda_{\alpha}(X)$.

Also $\|F(\mu)\|_* \leq \sup \{ |\mu|(K) p_K(f) : f \in C(X), \|f\|_\infty \leq 1 \}$

$= |\mu|(K) = \|\mu\|.$

Now we prove the reverse inequality, that is, $\|\mu\| \leq \|F(\mu)\|_*.$

Note $|\mu|(K) = \sup \{ \Sigma_i |\mu(A_i)| : \{A_i\} \text{ is a finite disjoint collection of } \mathcal{B} \text{ with } \bigcup A_i \subseteq K \}$. So given $\varepsilon > 0$, there exist $A_1, \ldots, A_n \in \mathcal{B}$ such that $A_i$'s are pairwise disjoint and $\Sigma_{i=1}^n |\mu|(A_i) > |\mu|(K) - \varepsilon$. Since $\mu$ is regular there exist compact sets $C_1$ and open sets $U_1$ such that $C_1 \subseteq A_1 \subseteq U_1$ and $|\mu|(U_1 \setminus C_1) < \varepsilon/n$ for $1 \leq i \leq n$. Since the compact subsets $C_i$'s are pairwise disjoint, pairwise disjoint open sets $V_1$ exist such that $C_i \subseteq V_i$. Now let
\( W_i = U_i \cap V_i \). Then \( C_i \cap (X \setminus W_i) = \emptyset \). Hence by Lemma 4.2, there exists a continuous function \( f_i : X + I \) such that \( f_i(C_i) = \{1\} \) and \( f_i(X \setminus W_i) = 0 \). Let \( a_i = \frac{\mu(A_i)}{\mu(A_i)} \) if \( \mu(A_i) \neq 0 \) and if \( |\mu(A_i)| = 0 \), let \( a_i = 0 \). Let \( f = \sum_{i=1}^{n} a_i f_i \). Since \( W_i \)'s are pairwise disjoint, \( \|f\|_{\infty} \leq 1 \).

Now \( \int f \ d\mu = \sum_{i=1}^{n} |\mu(A_i)| \)

\[
= \sum_{i=1}^{n} a_i \int f_i \ d\mu - \sum_{i=1}^{n} |\mu(A_i)|
\]

\[
= \sum_{i=1}^{n} a_i \left( f_i \left(W_i \setminus C_i\right) \right) \int d\mu
\]

\[
\leq \sum_{i=1}^{n} a_i |f_i| \left(W_i \setminus C_i\right) \int d\mu
\]

\[
< n \cdot \frac{\varepsilon}{n} + n \cdot \frac{\varepsilon}{n} = 2\varepsilon.
\]

So \( \|F(\mu)\|_{\ast} > \int f \ d\mu > \sum_{i=1}^{n} |\mu(A_i)| - 2\varepsilon > |\mu|(X) - 3\varepsilon = \|\mu\| - 3\varepsilon \). Therefore \( \|\mu\| - 3\varepsilon < \|F(\mu)\|_{\ast} \leq \|\mu\| \). Hence \( \|F(\mu)\|_{\ast} = \|\mu\| \), that is, \( F \) is an isometry.

Now we need to show that \( F \) is onto. Suppose \( \lambda \in \Lambda_{\alpha}(X) \). Then \( \lambda \) can be written as \( \lambda = \lambda^+ - \lambda^- \) where
λ^+, λ^- ∈ Λ^+_α(X). Now if λ has a compact support K belonging to α, then both λ^+ and λ^- have compact support K. To show F is onto, we try to get μ_1, μ_2 ∈ M^+_b,α(X) such that λ^+ = F(μ_1) and λ^- = F(μ_2). So

\[ λ = λ^+ - λ^- = F(μ_1) - F(μ_2) = F(μ_1 - μ_2) = F(μ) \]

where μ = μ_1 - μ_2 ∈ M^+_b,α(X). So we just need to consider λ^+. Define λ^+_K: C^*_∞(K) → IR as follows. For each f ∈ C^*_∞(K), choose an f_K ∈ C^*_α(X) such that f_K|_K = f. Then define λ^+_K(f) = λ^+(f_K). Since λ^+ is supported on K, λ^+_K is well-defined. Also since λ^+ is linear, so is λ^+_K. Finally λ^+_K is continuous since

\[ \sup \{|λ^+_K(f)|: f ∈ C^*(K), \|f\|_∞ ≤ 1\} = \sup \{|λ^+(f)|: f ∈ C^*(X), \|f\|_∞ ≤ 1\} = \|λ^+\|_* < ∞. \]

By the Riesz Representation Theorem (see [1]), there exists a μ_K ∈ M^+_b(K) such that λ^+_K(f) = ∫_K f dμ_K for all f ∈ C^*(K).

It only remains to show that an element μ_1 ∈ M^+_b(X) can be found such that μ_1(B) = μ_K(B ∩ K) for all B ∈ β. Then μ_1 would be supported on K so that μ_1 would be in M^+_b,α(X) and thus for each f ∈ C^*(X), λ^+(f) = λ^+_K(f|_K) = ∫_K f|_K dμ_K = ∫ f dμ_1 = F(μ_1)(f) which shows that λ^+ = F(μ_1).

First observe that because of Lemma 4.1, μ_1 is well-defined on β. So we only need to show that μ_1 is regular. Let B ∈ β and let ε > 0. Since μ_K is regular, there exists a compact subset C of K and an open subset U of K such that C ⊆ B ∩ K ⊆ U and μ_K(U \ C) < ε. Let V = U \ (X \ K) which is open in X. Then C ⊆ B ⊆ V and μ_1(V \ C) = μ_K((V \ C) ∩ K) = μ_K(U \ C) < ε. Therefore μ_1 is regular and is thus an element of M^+_b,α(X).
Note when \( \alpha = F(X) \) or \( K(X) \), the above theorem tells us what is exactly the measure-theoretic counterpart of \( \Lambda_p(X) \) or of \( \Lambda_k(X) \) respectively. Note that when \( \alpha = F(X) \), \( M_{b,\alpha}(X) \) is actually the linear space over \( \mathbb{R} \) generated by the set of Dirac's measures on \( X \). This fact explains why \( \Lambda_p(X) \) and \( \Lambda^+(X) \) cannot be complete because a limit of a Cauchy sequence in \( M_{b,p}(X) \) or in \( M_{b,\alpha}(X) \) may converge to a regular Borel measure on \( X \) with infinite support.

Now what is the measure-theoretic counterpart of \( \Lambda_\infty(X) \)? To answer this question, we introduce a new measure space. Let \( M_c(X) \) be the set of all bounded finitely additive regular measures defined on \( \Lambda_c \). Again \( M_c(X) \) is a normed linear space with the total variation norm. Let \( M^+_c(X) = \{ \mu \in M_c(X) : \mu \geq 0 \} \).

**Theorem 4.4.** If \( X \) is a normal and Hausdorff, then \((M_c(X), \| \cdot \|)\) is isometrically isomorphic to \((\Lambda_\infty(X), \| \cdot \|_*)\) while \( M^+_c(X) \) is identified with \( \Lambda^+_\infty(X) \) under this isometric isomorphism.

**Proof.** See [3], pages 78-83.

But what about the countable additivity of elements of \( M_c(X) \)? When \( X \) is countably compact, we have the following answer.

**Theorem 4.4.** If \( X \) is countably compact and if \( \mu \) is a bounded regular finitely additive measure defined on \( \Lambda_c \), then \( \mu \) is countably additive on \( \Lambda_c \), that is, \( \mu(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu(A_n) \) whenever \( (A_n) \) is a countable family of pairwise
disjoint sets from $A_c$ with union in $A_c$. Moreover $\mu$ has a regular countably additive extension to the $\sigma$-algebra $\mathcal{B}$ of Borel subsets of $X$.

Proof. See Theorem 3.11 in [7]. Also see [3].

Now the last theorem can be used to improve Theorem 4.4 to the following version.

Theorem 4.6. If $X$ is countably compact, normal and Hausdorff, then $M_\mathcal{B}(X)$ is isometrically isomorphic to $\Lambda_\infty(X)$ while $M_\mathcal{B}^+(X)$ is identified with $\Lambda_\infty^+(X)$ under this isometric isomorphism.

Since $\Lambda_{\alpha}(X) \subseteq \Lambda_\infty(X)$, for a countably compact, normal Hausdorff space, we have the following measure-theoretic counterpart of $\Lambda_{\alpha}(X)$.

Theorem 4.7. If $X$ is countably compact, normal and Hausdorff, then $M_{\mathcal{B},\alpha}(X)$ is isometrically isomorphic to $\Lambda_{\alpha}(X)$ while $M_{\mathcal{B},\alpha}^+(X)$ is identified with $\Lambda_{\alpha}^+(X)$ under this isometric isomorphism.

5. Density

The density $d(X)$ of a space $X$ is the smallest infinite cardinal number $m$ such that $X$ has a dense subset which has cardinality less than or equal to $m$. Now a space $X$ is separable if and only if $d(X) = \aleph_0$. If $X$ is a subspace of a metrizable space $Y$, then $d(X) \leq d(Y)$.

Theorem 5.1. For each space $X$, $d(\Lambda_{\alpha}^+(X)) = d(\Lambda_{\alpha}(X))$.

Proof. Use Theorem 3.1.
Corollary 5.2. $\Lambda_{\alpha}^{+}(X)$ is separable if and only if $\Lambda_{\alpha}(X)$ is separable.

For each $x \in X$, define the evaluation function at $x$, 
$\phi_{x}: \mathcal{C}^{*}(X) \rightarrow \mathbb{R}$ by taking $\phi_{x}(f) = f(x)$ for each $f \in \mathcal{C}^{*}(X)$. Now $\phi_{x}$ is a positive linear functional on $\mathcal{C}^{*}(X)$ which is supported on $\{x\}$. Now if $\{x\} \in \alpha$, then by Lemma 2.1 $\phi_{x} \in \Lambda_{\alpha}^{+}(X)$.

For the remainder of this section, the notation $|X|$ stands for the cardinality of $X$.

Theorem 5.3. Suppose $F(X) \subseteq \alpha$. Then $|X| \leq d(\Lambda_{\alpha}^{+}(X))$.

Proof. Since $F(X) \subseteq \alpha$, $\phi_{x} \in \Lambda_{\alpha}^{+}(X)$ for all $x \in X$. Define the evaluation function $\phi: X \rightarrow \Lambda_{\alpha}^{+}(X)$ by taking $\phi(x) = \phi_{x}$. Since $\mathcal{C}^{*}(X)$ separate points, then $\phi$ is one-to-one. Therefore $|\phi(X)| = |X|$. Now let $x$ and $y$ be distinct points of $X$. Then $d_{\sigma}(\phi_{x}, \phi_{y}) = \|\phi_{x} - \phi_{y}\| = \sup \{|\phi_{x}(f) - \phi_{y}(f)|: f \in \mathcal{C}^{*}(X), \|f\|_{\infty} \leq 1\} = \sup \{|f(x) - f(y)|: f \in \mathcal{C}^{*}(X), \|f\|_{\infty} \leq 1\} > 1$. So $\phi(X)$ is a discrete subset of $\Lambda_{\alpha}^{+}(X)$ and hence $|\phi(X)| \leq d(\phi(X))$. Therefore $|X| = |\phi(X)| \leq d(\phi(X)) \leq d(\Lambda_{\alpha}^{+}(X))$.

Corollary 5.4. Suppose $F(X) \subseteq \alpha$ and $\Lambda_{\alpha}^{+}(X)$ is separable. Then $X$ is countable.

In order to establish a more general theorem on separability of $\Lambda_{\alpha}^{+}(X)$, we need to discuss the separability of $M_{b,\alpha}(X)$. Note that the proof of Theorem 3.3 in [8] actually shows that if $X$ is countable, then $M_{b}(X)$ is
separable. So when \( X \) is countable, \( M_{b,\alpha}(X) \) and \( M_{b,\alpha}^+(X) \) are also separable.

\textbf{Theorem 5.5.} Suppose \( F(X) \subseteq \alpha \subseteq K(X) \). Then \( \Lambda_\alpha^+(X) \) is separable if and only if \( X \) is countable.

\textit{Proof.} Suppose \( \Lambda_\alpha^+(X) \) is separable. Then by Corollary 5.4, \( X \) is countable. Conversely, let \( X \) be countable. Now since \( \alpha \subseteq K(X) \), by Theorem 4.3, \( \Lambda_\alpha^+(X) \) is isomorphic to \( M_{b,\alpha}^+(X) \). So \( d(\Lambda_\alpha^+(X)) = d(M_{b,\alpha}^+(X)) \). Hence \( \Lambda_\alpha^+(X) \) is separable.

Lastly, we talk about the separability of \( \Lambda_\infty(X) \).
Note that Theorem 5.1 gives us \( d(\Lambda_\infty^+(X)) = d(\Lambda_\infty(X)) \). This means that \( \Lambda_\infty(X) \) is separable if and only if \( \Lambda_\alpha^+(X) \) is separable.

\textbf{Theorem 5.6.} \( \Lambda_\infty(X) \) is separable if and only if \( X \) is compact and countable.

\textit{Proof.} If \( \Lambda_\infty(X) \) is separable, then \( \Lambda_k(X) \) is separable and so \( X \) is countable. Again, since \( \Lambda_\infty(X) \) is the conjugate space of the normed linear space \( C_\infty^*(X) \), \( C_\infty^*(X) \) is separable. But this implies that \( X \) is compact (see [9], page 54).
Conversely, let \( X \) be compact and countable. Since \( X \) is compact, \( C_k^*(X) = C_\infty^*(X) \) and consequently \( \Lambda_\infty(X) = \Lambda_k(X) \).
But \( X \) is countable and so \( \Lambda_k(X) \) is separable. Hence \( \Lambda_\infty(X) \) is separable.

\textbf{References}


Virginia Polytechnic Institute and State University

Blacksburg, Virginia 24061