OPEN MAPS ON RIMCOMPACT SPACES

by

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All spaces in this paper will be completely regular and Hausdorff. A space $X$ is rimcompact if $X$ has a base of open sets with compact boundaries, almost rimcompact if $X$ has a compactification $KX$ in which each point of $KX \setminus X$ has a base in $KX$ of open sets whose boundaries lie in $X$, and a $0$-space if $X$ has a compactification with zero-dimensional remainder.

As mentioned in [Ku], Misic' has pointed out that the property of rimcompactness is preserved under mappings that are simultaneously open and closed, denoted open-closed. The argument is straightforward—if $f : X \to Y$ is open-closed and $U$ is open in $X$ with compact boundary, then $f[U]$ is an open subset of $Y$ having compact boundary. According to [Ku], this statement also holds for monotone open maps, so that monotone open maps preserve the property of rimcompactness. This last result is extended in [Di3] to the case in which the space $X$ is almost rimcompact or a $0$-space. In this paper we indicate that monotone open maps and open-closed maps possess a more general property sufficient for this extension.

An open set $U$ of $βX$ is $CI$ in $βX$ (denoting clopen at infinity) if $U \cap (βX \setminus X)$ is clopen in $βX \setminus X$. An open set $U$ of $X$ is $π$-open in $X$ if $\text{bd}_X U$ is compact. For a map $f : X \to Y$, $f^β$ will denote the extension map from $βX$ into $βY$. 

Definition 1. An open map $f: X \to Y$ is CI preserving if whenever $U$ is a CI open subset of $\beta X$, $f^\beta[U]$ is a CI open subset of $\beta Y$.

As in [Di3], for a space $X$ and $p \in \beta X$, $K_p$ will denote $\cap \{\beta X \cup U: U \text{ is CI open in } X, p \not\in U\}$. According to 2.2 and 2.3 of [Di3], for $p \in \beta X$, $K_p$ is a connected compact subset of $\beta X$. If $K_p \subseteq \beta X \setminus X$, then $K_p$ is the quasicomponent of $p$ in $\beta X \setminus X$ and has a base of CI open sets in $\beta X$. If $X$ is a 0-space, for $p \in \beta X \setminus X$, $K_p \subseteq \beta X \setminus X$.

Theorem 1. If $f: X \to Y$ is CI preserving and $X$ is a 0-space, $Y$ is a 0-space.

Proof. We first show that if $f: X \to Y$ is CI preserving and $X$ is a 0-space, then for $p \in \beta Y \setminus Y$, $K_p \subseteq \beta Y \setminus Y$.

Choose $p \in \beta Y \setminus Y$, $y \in Y$ and $x \in f^+(y) \cap X$. For each $z \in f^\beta+(p)$, $z \in \beta X \setminus X$ so that $K_z \subseteq \beta X \setminus X$ and $x \not\in K_z$. There is a CI open set $V_z$ of $\beta X$ such that $x \in V_z$ while $z \not\in V_z$.

For each $z$, $(\beta X \setminus V_z) \cap (\beta X \setminus X)$ is clopen in $\beta X \setminus X$. Then $f^\beta+(p) \subseteq \bigcup \{(\beta X \setminus V_z) \cap (\beta X \setminus X): z \in f^\beta+(p)\}$, thus there is a finite subset $\{z_i: i = 1 \text{ to } n\}$ of $f^\beta+(p)$ such that $f^\beta+(p) \subseteq \bigcup_{i=1}^{n} (\beta X \setminus V_{z_i}) \cap (\beta X \setminus X)$. Let $V = \cap_{i=1}^{n} V_{z_i}$.

Then $x \in V$, $f^\beta+(p) \cap V = \emptyset$ and $V$ is CI open in $\beta X$. It follows that $y = f(x) \in f^\beta[V]$ which is CI and open in $\beta Y$, while $p \not\in f^\beta[V]$. Thus $K_p \subseteq \beta Y \setminus Y$.

The theorem now follows from the proof of 2.6 of [Di3].
Theorem 2.6 of [Di₃] is stated for \( f: X \rightarrow Y \) monotone open. However, a careful reading of the proof will indicate that the property of the function \( f \) actually used is that such a function is CI preserving.

To work with rimcompact and almost rimcompact spaces we need to look at images of \( \pi \)-open sets. In the following, if \( U \) is open in \( \beta X \), \( \beta X \setminus U \) will denote \( \beta X \setminus \text{cl}_{\beta X} (X \setminus U) \), the largest open subset of \( \beta X \) whose intersection with \( X \) is \( U \).

**Lemma 1.** If \( f: X \rightarrow Y \) is CI preserving and \( U \) is \( \pi \)-open in \( X \), then \( f[U] \) is \( \pi \)-open in \( Y \) and \( \text{cl}_Y f[U] = f[\text{cl}_X U] \).

**Proof.** The set \( f[U] \) is clearly open in \( Y \). For any continuous \( f: X \rightarrow Y \), \( f^\beta \) is closed, so that \( \text{cl}_Y f[U] = f^\beta [\text{cl}_X U] \). Now \( U \) is \( \pi \)-open in \( X \), hence \( \text{cl}_X U \setminus \beta X \setminus U = \text{bd}_X \beta X \setminus U = \text{bd}_X U \subseteq X \) (see [Sk] or [Di₃]).

Since \( f \) is CI preserving, \( \text{bd}_X f^\beta [\text{cl}_X U] \subseteq \text{cl}_Y f^\beta \beta X \setminus \text{bd}_X U \subseteq \text{cl}_Y \beta X \setminus \text{bd}_X U \subseteq f^\beta [\text{bd}_X U] \subseteq Y \). Finally, \( \text{bd}_Y f[U] \subseteq \text{bd}_X f^\beta [\text{cl}_X U] \subseteq \text{bd}_X f[U] \cup f[\text{bd}_X U] = \text{cl}_Y f[U] \subseteq f[\text{cl}_X U] \) and \( \text{bd}_Y f[U] \) is compact.

Example 1 will indicate that a CI preserving map need not be closed on all closed sets, even if the space \( X \) is rimcompact, so that the \( \pi \)-open sets form a base.

**Corollary 1.** Suppose that \( f: X \rightarrow Y \) is CI preserving, and that \( X \) is almost rimcompact (rimcompact). Then \( Y \) is almost rimcompact (rimcompact).
Proof. Suppose first that $X$ is almost rimcompact. It follows from Theorem 1 that $Y$ is a 0-space. According to 2.7 of [Di₁], $Y$ is almost rimcompact if and only if each $y \in Y$ has the property (*): there is a compact set $K_y$ of $Y$ such that if $F$ is closed in $Y$ and $F \cap K_y = \emptyset$, there is a $\pi$-open subset $V$ of $Y$ with $y \in V$ and $cl_y V \cap F = \emptyset$. Since $X$ is almost rimcompact, it also follows from 2.7 of [Di₁] that each $x \in X$ has property (*).

Suppose that $y \in Y$; choose $x \in f^+(y)$ and $K_x$ witnessing the fact that $x$ has property (*). Let $K_y = f[K_x]$, and suppose that $F$ is closed in $Y$ with $F \cap K_y = \emptyset$. Then $f^+[F]$ is a closed subset of $X$ with $f^+[F] \cap K_x = \emptyset$. Choose $V$ to be $\pi$-open in $X$ with $x \in V$ and $cl_x V \cap f^+[F] = \emptyset$. Then $y \in f[V]$ which is $\pi$-open in $Y$ with $F \cap cl_y f[V] = F \cap f[cl_x V] = \emptyset$.

If $X$ is rimcompact, we can choose $K_x = \{x\}$ in the above argument. Then $K_y = \{y\}$, indicating that $Y$ has a base of $\pi$-open sets.

In the above argument it is not necessary that each $x \in X$ have property (*) in either the rimcompact or almost rimcompact case. In [Di₃], pointwise definitions of rimcompactness and almost rimcompactness are made. (That $x \in X$ have property (*) is one of two conditions defining almost rimcompactness of $X$ at $x$.) The use of Theorem 1 of this paper is no longer valid under the hypotheses of the next result, but arguments similar to those in Corollary 4 above and 2.8 and 2.9 of [Di₃] yield the following:
Theorem 2. Suppose that \( f: X \to Y \) is CI preserving, and that for \( y \in Y \), \( f^+(y) \) contains a point at which \( X \) is almost rimcompact (rimcompact). Then \( Y \) is almost rimcompact (rimcompact).

Finally, we indicate the existence of nontrivial CI preserving maps. The next result generalizes 2.1 of [Di3] by removing the hypothesis that \( X \) be a 0-space.

Theorem 3. If \( f: X \to Y \) is monotone and open, then \( f \) is CI preserving.

Proof. Suppose that \( U \) is open and CI in \( \beta X \), and \( p \in (\beta Y \setminus Y) \cap f^\beta[U] \). Then \( f^{\beta^+}(p) \subseteq \beta X \setminus X \) and \( f^{\beta^+}(p) \cap U \neq \emptyset \). According to 4.7 of [Di4], \( f^\beta \) is monotone; since \( U \cap (\beta X \setminus X) \) is clopen in \( \beta X \setminus X \), \( f^{\beta^+}(p) \subseteq U \). Then \( f^{\beta^+}[f^\beta[U] \cap (\beta Y \setminus Y)] = U \cap f^{\beta^+}[\beta Y \setminus Y] \), thus \( f^\beta[U] \cap (\beta Y \setminus Y) \) is clopen in \( \beta Y \setminus Y \). Since \( f^\beta \) is closed, \( p \in \text{int}_{\beta Y} f^\beta[U] \).

It remains to show that \( f^\beta[U] \cap Y \subseteq \text{int}_{\beta Y} f^\beta[U] \). We first show that \( f^\beta[U] \cap Y = f[U \cap X] \). If \( p \in [f^\beta[U] \cap Y] \cap f[U \cap X] \), then \( f^+(p) \subseteq X \setminus U \), so that \( \text{cl}_{\beta X} f^+(p) \cap U = \emptyset \). It follows that \( f^{\beta^+}(p) \cap U = f^{\beta^+}(p) \cap U \cap (\beta X \setminus X) \) and is open in \( f^{\beta^+}(p) \). Since \( (\beta X \setminus X) \setminus U \) is clopen in \( \beta X \setminus X \), there is an open set \( W \) of \( \beta X \) such that \( W \cap (\beta X \setminus X) = (\beta X \setminus X) \setminus U \), thus \( W \cap U \subseteq X \). Then \( f^{\beta^+}(p) \cap U = f^{\beta^+}(p) \setminus W \) and so is closed in \( f^{\beta^+}(p) \). That is, \( f^{\beta^+}(p) \cap U \) is clopen in the connected set \( f^{\beta^+}(p) \), a contradiction.

Choose \( x \in f^+(p) \cap U \) and \( W \) open in \( \beta X \) with \( x \in W \subseteq \text{cl}_{\beta X} W \subseteq U \). Since \( f[W \cap X] \) is open in \( Y \),
\( p \in \text{int}_{\beta Y} \text{cl}_{\beta Y} f[\omega \cap X] \subseteq \text{cl}_{\beta Y} f[\omega \cap X] \subseteq f^\beta [\text{cl}_{\beta X} \omega] \subseteq f^\beta [U]. \)

That is, \( p \in \text{int}_{\beta Y} f^\beta [U]. \)

In 3.4 of [Di2], a rimcompact space \( X, \) nonrimcompact space \( Y \) and monotone closed map \( f: X \rightarrow Y \) are constructed, indicating that monotone closed maps are not CI preserving.

**Theorem 4.** Suppose that \( f \) and \( f^\beta \) are open. Then \( f \) is CI preserving.

**Proof.** If \( U \) is open and CI in \( \beta X, \) then \( f^\beta [U] \) is an open subset of \( \beta Y. \) Also, \( U \cap f^\beta [\beta Y \setminus Y] \) is clopen in \( f^\beta [\beta Y \setminus Y]. \) Since \( f^\beta \) restricted to \( f^\beta [\beta Y \setminus Y] \) is a closed map, \( f^\beta [U] \cap (\beta Y \setminus Y) \) is clopen in \( \beta Y \setminus Y. \)

A map \( f: X \rightarrow Y \) is a WZ map if \( \text{cl}_{\beta X} f^+(y) = f^\beta+(y) \) for each \( y \in Y. \) As pointed out in 1.1 of [Is], a closed map is a WZ map. Theorem 4.4 of the same paper states that if \( f: X \rightarrow Y \) is a WZ map, then \( f^\beta \) is open if and only if \( f \) is open. Thus we have the following:

**Corollary 2.** If \( f: X \rightarrow Y \) is open and WZ, then \( f \) is CI preserving.

**Corollary 3.** If \( f: X \rightarrow Y \) is open-closed, then \( f \) is CI preserving.

The next example indicates that, as mentioned earlier, a CI preserving map on a space \( X \) need not be closed, even if the \( \pi \)-open sets form a base for \( X. \) It also indicates that an open WZ map need not be closed.
Example 1. Let $X$ be the space of countable ordinals, and $Y$ the space $X$ in addition to the first uncountable ordinal. As discussed in 3.10.16 of [En], the projection map $\pi_Y: X \times Y \to Y$ is not closed. Since $X \times Y$ is pseudo-compact, $\beta(X \times Y) = \beta X \times Y$ (Theorems 1 and 4 of [Gl]).

The extension of $\pi_Y$ over $\beta(X \times Y)$ is clearly the projection map $\pi_{\beta X}: \beta X \times Y \to Y$. Then $c_{\beta X \times Y} \pi_Y^+(y) = \beta X \times \{y\} = \pi_{\beta X}^+(y)$, so that $\pi_Y$ is an open $WZ$ map.

References


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