EACH MAP FROM THE CANTOR SET TO THE PSEUDO-ARC IS NULL PSEUDO-HOMOTOPIE

by

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1. Introduction

A compact connected metric space is called a continuum. K. Kuperberg posed a problem whether the pseudo-arc is pseudo-contractible (University of Houston Problem Book, Problem 31). See below for the definition. In connection with this problem, D. Bellamy [1] constructed a map from the Cantor set onto the pseudo-arc which is null pseudo-homotopic. He also asked ([1], Question 1) whether each map from the Cantor set onto the pseudo-arc is null pseudo-homotopic. The purpose of this paper is to answer the above question in the affirmative. More precisely, we show that each map from the Cantor set to the pseudo-arc (not necessarily onto) is null pseudo-homotopic. Moreover, the parameter space can be taken to be the pseudo-arc.

2. Preliminaries

Definition 1. Let $X$ and $Y$ be continua and $f, g: X \to Y$ be maps. We say that $f$ and $g$ are pseudo-homotopic if there exist a continuum $Z$, points $a, b \in Z$ and a map $H: X \times Z \to Y$ such that $H(x, a) = f(x), H(x, b) = g(x)$ for each $x \in X$. The continuum $Z$ is called the parameter space of a pseudo-homotopy $H$. 

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A map which is pseudo-homotopic to a constant map is said to be null pseudo-homotopic. If \( \text{id}_X : X \to X \) is null pseudo-homotopic, then we say that \( X \) is pseudo-contractible.

Definition 2. 1) Let \( U = \{U_1, \ldots, U_n\} \) be a collection of sets. The collection \( U \) is called a chain provided \( U_i \cap U_j \neq \emptyset \) if and only if \( |i-j| \leq 1 \).

2) A function \( f : \{1, \ldots, m\} \to \{1, \ldots, n\} \) is called a pattern if \( |f(i) - f(i+1)| \leq 1 \) for each \( i = 1, \ldots, m-1 \).

3) Let \( U = \{U_1, \ldots, U_m\} \) and \( V = \{V_1, \ldots, V_n\} \) be chains, and \( f : \{1, \ldots, m\} \to \{1, \ldots, n\} \) be a pattern. We say that \( U \) follows \( f \) in \( V \) if \( U_i \subseteq V_{f(i)} \) for each \( i = 1, \ldots, m \). In this case, a function \( \overline{f} : U \to V \) is defined by \( \overline{f}(U_i) = V_{f(i)} \). We will identify \( f \) and \( \overline{f} \).

4) Let \( U = \{U_1, \ldots, U_n\} \) be a chain cover of a continuum. The links \( U_1 \) and \( U_n \) are denoted by first \( U \) and last \( U \) respectively. For each \( k (1 \leq k \leq n) \), \( i(U_k) \) is defined by \( U_k = \text{cl}(\bigcup_{j \neq k} U_j) \).

Definition 3. Let \( X \) be a continuum.

1) \( X \) is said to be arc-like if, for each \( \varepsilon > 0 \), there exists a chain cover \( U \) of \( X \) such that \( \text{mesh} U < \varepsilon \).

2) \( X \) is said to be hereditarily indecomposable if no subcontinuum of \( X \) can be represented as the union of two of its proper subcontinua.

3) Hereditarily indecomposable arc-like continuum is topologically unique ([3] and [6]), which is called
the pseudo-arc. Throughout this paper, the pseudo-arc is denoted by $P$.

4) Let $p$ and $q$ be points of $X$. $X$ is said to be irreducible between $p$ and $q$, if $X$ contains no proper subcontinuum which contains both of $p$ and $q$.

The following theorem is well known and will be used for the proof.

Theorem 4 ([2] and [5]). Let $C = \{C_1, \ldots, C_n\}$ be a chain cover of $P$ and $x \in \iota(C_1)$, $y \in \iota(C_n)$. Suppose that $P$ is irreducible between $x$ and $y$. Then for each pattern $f: \{1, \ldots, m\} \to \{1, \ldots, n\}$ with $f(1) = 1$ and $f(m) = n$, there exists a chain cover $D = \{D_1, \ldots, D_m\}$ which follows $f$ in $C$, and $x \in \iota(D_1)$, $y \in \iota(D_m)$.

3. The Main Theorem

Our main theorem is

Theorem 5. Each map from the Cantor set to the pseudo-arc is null pseudo-homotopic. Furthermore, we can take the parameter space of the pseudo-homotopy as the pseudo-arc.

In the rest of this paper, $C$ denote the Cantor set.

The following theorem is the key step.

Proposition 6. Suppose that a map $f: C \to P$ satisfies the following condition:
there exists a point \( a_0 \in P \) such that \( P \) is irreducible between \( a_0 \) and \( y \), for each \( y \in f(C) \).

Then \( f \) is pseudo-homotopic to a constant map with the parameter space \( P \).

**Proof.** Suppose that \( P \) is irreducible between \( x_0 \) and \( y_0 \). We can take a sequence \( (\mathcal{U}_n)_{n \geq 0} \) of open covers of \( C \) as follows:

a) Each \( \mathcal{U}_n \) is a mutually disjoint clopen cover of \( C \).

b) \( \mathcal{U}_{n+1} \) is a refinement of \( \mathcal{U}_n \) for each \( n \).

c) mesh \( \mathcal{U}_n \to 0 \) as \( n \to \infty \).

**Step 1.** For each \( x \in C \), there exists a chain cover \( V_x \) of \( P \) such that

1-1) \( f(x) \in i(\text{first } V_x) \) and \( a_0 \in i(\text{last } V_x) \).

1-2) mesh \( V_x < 1/4 \) ([2], [4]).

By c) and the continuity of \( f \), we can take an integer \( n(x) > 0 \) such that

1-3) \( f(V_{n(x)}(x)) \subset i(\text{first } V_x) \),

where, \( V_{n(x)}(x) \) denotes the unique member of \( \mathcal{U}_n(x) \) which contains \( x \).

The collection \( \{V_{n(x)}(x) | x \in C \} \) forms an open cover of \( C \), so we can take finitely many points \( x_1, \ldots, x_r \in C \) such that \( C = \bigcup_{i=1}^{r} V_{n(x_i)}(x_i) \). Define \( n_1 \) as

1-4) \( n_1 = \max \{n(x_i) | 1 \leq i \leq r \} \).

Then noticing b), we have

1-5) for each \( D \in \mathcal{U}_{n_1} \), there exists a chain cover \( V_D^{1} \)

such that \( f(D) \subset i(\text{first } V_D^{1}) \) and \( a_0 \in i(\text{last } V_D^{1}) \).
For each member $D$ of $\mathcal{D}_{n_1}$, we define a chain cover $U_D^1$ of $P$ as follows.

1-6) (The number of links of $U_D^1$) = (The number of links of $V_D^1$)

1-7) $x_0 \in i$(first $U_D^1$) and $y_0 \in i$(last $U_D^1$).

Now we have an open cover $D \times U_D^1$ of $D \times P$, for each $D \in \mathcal{D}_{n_1}$.

Step 2. Fix a member $D_1$ of $\mathcal{D}_{n_1}$. For each $x \in D_1$, we can take a chain cover $V_x^2$ of $P$ such that

2-1) $f(x) \in i$(first $V_x^2$) and $a_0 \in i$(last $V_x^2$).

2-2) mesh $V_x^2 < 1/8$ and $V_x^2$ is a closure refinement of $V_{D_1}^1$ (that is, for each $V \in V_x^2$, there exists $U \in V_{D_1}^1$ such that $cl(V) \subseteq U$).

Again by c), there exists an integer $m(x) > 0$ such that

2-3) $f(w_m(x)) \subseteq i$(first $V_x^2$).

The collection $\{w_m(x) | x \in D_1\}$ forms an open cover of $D_1$, so there exist finitely many points $y_1, \ldots, y_s \in D_1$ such that $D_1 = \bigcup_{j=1}^{s} w_m(y_j) (y_j)$.

Repeating these processes for all members of $\mathcal{D}_{n_1}$, we obtain finitely many points $y_1, \ldots, y_t$ and chain covers $V_{y_1}^2, \ldots, V_{y_t}^2$. Define $n_2$ as

2-4) $n_2 = \max \{m(y_j) | 1 \leq j \leq t\}$.

Then we have
2-5) for each $D_2 \in \mathcal{D}_{n_2}$, there exists a chain cover $V^2_{D_2}$ such that $f(D_2) \subseteq i(\text{first } V^2_{D_2})$ and $a_0 \in i(\text{last } V^2_{D_2})$.

Next, we define a pattern as follows. For each $D_2 \in \mathcal{D}_{n_2}$, take the unique $D_1 \in \mathcal{D}_{n_1}$ which contains $D_2$. Then by the choice of $V^2_{D_2}$ (2-2), $V^2_{D_2}$ is a closure refinement of $V^1_{D_1}$. So we can find a pattern $f_{D_2D_1} : V^2_{D_2} + V^1_{D_1}$ such that

$\begin{align*}
&f_{D_2D_1} \text{ (first } V^2_{D_2}) = \text{first } V^1_{D_1} \text{ and } \\
&f_{D_2D_1} \text{ (last } V^2_{D_2}) = \text{last } V^1_{D_1}. \text{ (Recall the remark in Definition 2).}
\end{align*}$

Applying Theorem 4, there exists a chain cover $U^2_{D_2}$ of $P$ such that

2-6) $U^2_{D_2}$ follows $f_{D_2D_1}$ in $U^1_{D_1}$.

2-7) $x_0 \in i(\text{first } U^2_{D_2})$ and $y_0 \in i(\text{last } U^2_{D_2})$.

Now, we have a covering $D_2 \times U^2_{D_2}$ of $D_2 \times P$, for each $D_2 \in \mathcal{D}_{n_2}$.

Step 3. Continuing these processes, we obtain a subsequence $(n_k)_{k>1}$ satisfying the following conditions.

3-1) $x_0 \in i(\text{first } U^k_{D_k})$ and $y_0 \in i(\text{last } U^k_{D_k})$.

3-2) $f(D_k) \subseteq i(\text{first } U^k_{D_k})$ and $a_0 \in i(\text{last } U^k_{D_k})$. 


3-3) For each $D_k \supset D_{k+1}$ ($D_\alpha \in \mathcal{D}_{\alpha}$, $\alpha = k, k+1$), there exists a pattern $f_{D_{k+1}D_k}$ such that $u_{D_{k+1}}^{k+1}$ ($v_{D_{k+1}D_k}^{k+1}$ resp.) follows $f_{D_{k+1}D_k}$ in $u_{D_k}^k$ ($v_{D_k}^k$ resp.).

3-4) $f_{D_{k+1}D_k}$ (first $u_{D_k}^k$) = first $u_{D_k}^k$, and $f_{D_{k+1}D_k}$ (last $u_{D_k}^k$) = last $u_{D_k}^k$.

The same conditions hold for $v_{D_{k+1}D_k}^k$ and $v_{D_k}^k$.

3-5) mesh $\nu_{D_k}^k < 1/2^{k+1}$ for each $k \geq 1$.

There are then more and more chains, both $V$'s and $U$'s at each stage than there were before. Each chain at the $k$-level has several different refining chains at $(k+1)$-level.

Finally, we define $H: \mathcal{C} \times \mathcal{P} \to \mathcal{P}$ as follows. For each $x \in \mathcal{C}$, there exists the unique sequence $D_1(x) \supset D_2(x) \supset \ldots$ with $D_k(x) \in \mathcal{D}_{n_k}$ such that $\{x\} = \bigcap_{k \geq 1} D_k(x)$.

Then we have two sequences $\{u_{D_k(x)}^k\}_{k \geq 1}$ and $\{v_{D_k(x)}^k\}_{k \geq 1}$ of chain covers of $\mathcal{P}$. By the standard method of constructing a map between the pseudo-arcs, we have a map $H|_{x \times \mathcal{P}}: x \times \mathcal{P} \to \mathcal{P}$ such that

3-6) $H(x \times u_{D_k(x)}^k(i)) \subseteq \text{st}(v_{D_k(x)}^k(i), u_{D_k(x)}^k)$ for each $i = 1, \ldots, k.$

Notice the following.

3-7) If $x, y \in D_k \in \mathcal{D}_{n_k}$, then $u_{D_i(x)}^i = u_{D_i(y)}^i$ and $v_{D_i(x)}^i = v_{D_i(y)}^i$ for each $i = 1, \ldots, k$. 
Using this fact, it is easy to see that the map $H$ defined as above is continuous and $H(x, x_0) = f(x)$, $H(x, y_0) = a_0$ for each $x \in C$. This completes the proof.

Proof of Theorem 5.

Let $f: C \to P$ be a map. Take a nondegenerate proper subcontinuum $Q$ of $P$. By [3], $Q$ is a retract of $P$. Fix a retraction $r: P \to Q$ and a homeomorphism $h: P \to Q$. Fix a point $a_0$ of $Q$ which lies in a different component from $Q$. Applying Proposition 6 to $h \circ f: C \to Q$ and $a_0$, we have a map $H: C \times P \to P$ and points $x_0$ and $y_0 \in P$ such that $H|C \times x_0 = h \circ f$ and $H|C \times y_0 = a_0$. Define $F: C \times P \to P$ as $F = h^{-1} \circ r \circ H$. Then $F|C \times x_0 = f$ and $F|C \times y_0 = h^{-1}(a_0)$. This completes the proof of Theorem 5.

Corollary 7. Any Cantor set in the pseudo-arc $P$ is pseudo-contractible in $P$.

References


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